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THE FOUNDATIONS OF
EINSTEIN'S THEORY OF GRAVITATION

ERWIN FREUNDLICH

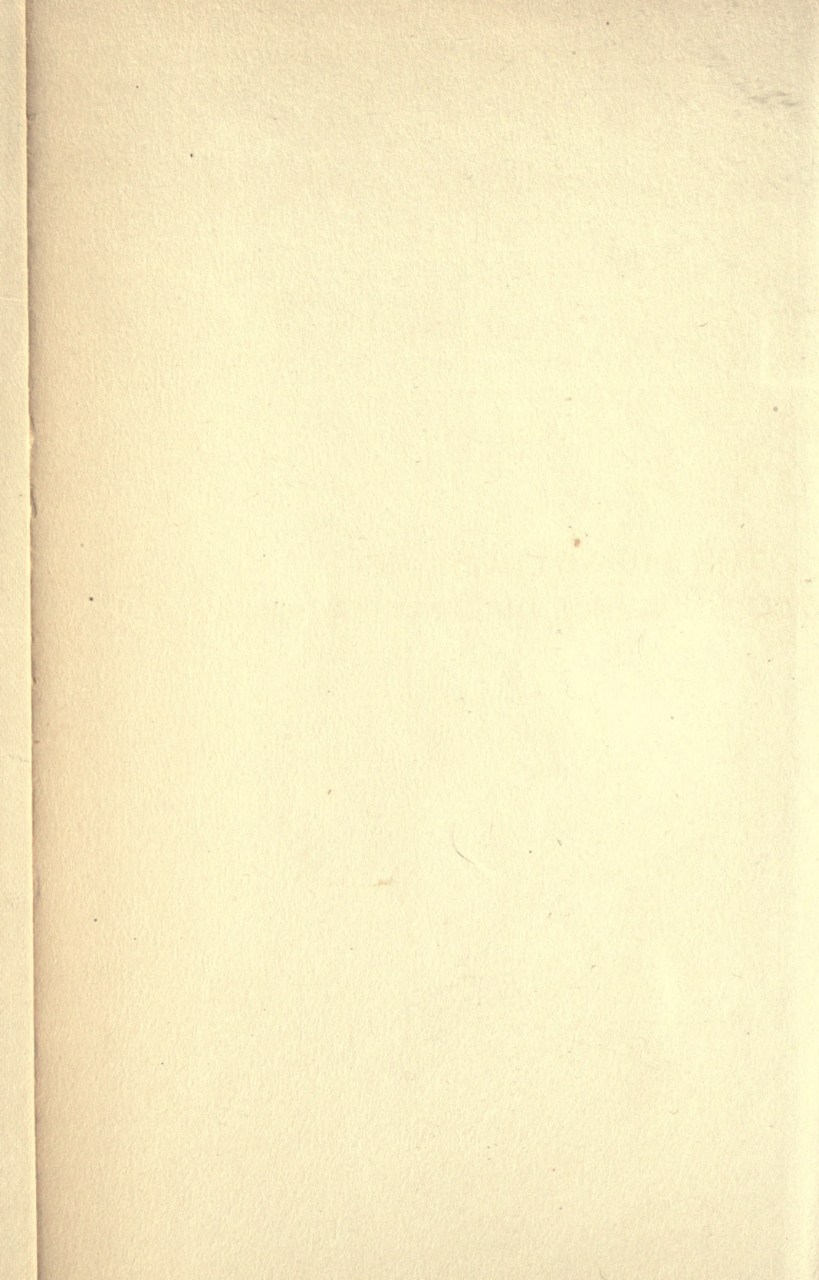


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**THE FOUNDATIONS OF
EINSTEIN'S THEORY OF GRAVITATION**

THE FOUNDATIONS OF EINSTEIN'S THEORY OF GRAVITATION

BY

ERWIN FREUNDLICH

DIRECTOR OF THE EINSTEIN TOWER

WITH A PREFACE BY

ALBERT EINSTEIN

TRANSLATED FROM THE FOURTH GERMAN EDITION, WITH TWO ESSAYS, BY

HENRY L. BROSE

CHRIST CHURCH, OXFORD

WITH AN INTRODUCTION BY

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WITH FIVE DIAGRAMMS

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PREFACE

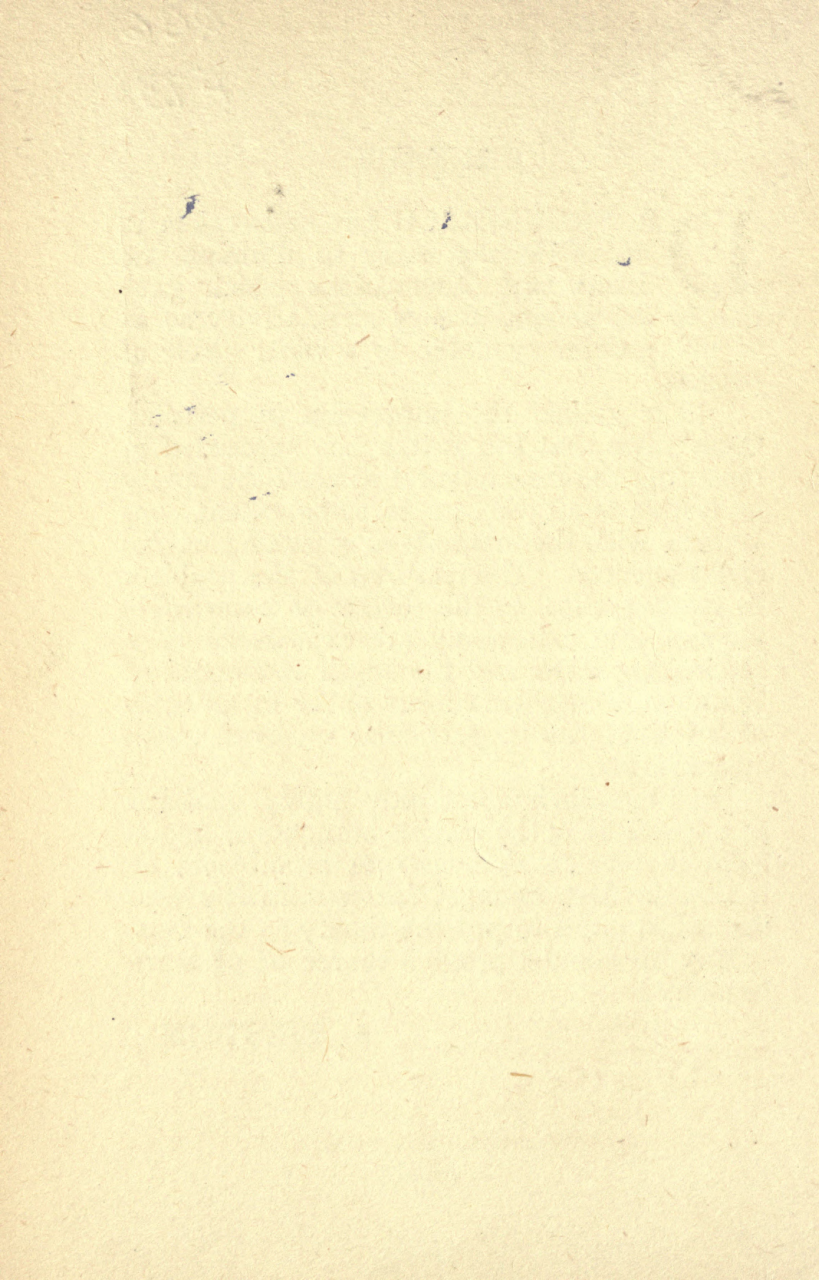
DR. FREUNDLICH has undertaken in the following essay to illumine the ideas and observations which gave rise to the general theory of relativity so as to make them available to a wider circle of readers.

I have gained the impression in perusing these pages that the author has succeeded in rendering the fundamental ideas of the theory accessible to all who are to some extent conversant with the methods of reasoning of the exact sciences. The relations of the problem to mathematics, to the theory of knowledge, physics and astronomy are expounded in a fascinating style, and the depth of thought of Riemann, a mathematician so far in advance of his time, has in particular received warm appreciation.

Dr. Freundlich is not only highly qualified as a specialist in the various branches of knowledge involved to demonstrate the subject ; he is also the first amongst fellow-scientists who has taken pains to put the theory to the test.

May his booklet prove a source of pleasure to many !

A. EINSTEIN



INTRODUCTION

THE Universe is limited by the properties of light. Until half a century ago it was strictly true that we depended upon our eyes for all our knowledge of the universe, which extended no further than we could see. Even the invention of the telescope did not disturb this proposition, but it is otherwise with the invention of the photographic plate. It is now conceivable that a blind man, by taking photographs and rendering their records in some way decipherable by his fingers, could investigate the universe; but still it would remain true, that all his knowledge of anything outside the earth would be derived from the use of light and would therefore be limited by its properties. On this little earth there is, indeed, a tiny corner of the universe accessible to other senses: but feeling and taste act only at those minute distances which separate particles of matter when "in contact:" smell ranges over, at the utmost, a mile or two; and the greatest distance which sound is ever known to have travelled (when Krakatoa exploded in 1883) is but a few thousand miles—a mere fraction of the earth's girdle. The scale of phenomena manifested through agencies other than light is so small that we are unlikely to reach any noteworthy precision by their study.

Few people who are not astronomers have spent much thought on the limitations introduced by the

news agency to which we are so profoundly indebted. Light comes speedily but has far to travel, and some of the news is thousands of years old before we get it. Hence our universe is not co-existent : the part close around us belongs to the peaceful present, but the nearest star is still in the midst of the late War, for our news of him is three years old ; other stars are Elizabethan, others belong to the time of the Pharaohs ; and we have alongside our modern civilization yet others of prehistoric date. The electric telegraph has accustomed us to a world in which the news is approximately of even date : but our forefathers must have been better able, from their daily experience of getting news many months old, to realize the unequal age of the universe we know. Nowadays the inequality is almost entirely the concern of the astronomer, and even he often neglects or forgets it. But when fundamental issues are at stake, the time taken by the messenger is an essential part of the discussion, and we must be careful to take account of it, with the utmost precision.

Our knowledge that light had a finite velocity followed on the invention of the telescope and the discovery of Jupiter's satellites : the news of their eclipses came late at times and these times were identified as those when Jupiter was unusually far away from us. But the full consequences of the discovery were not realized at first. One such consequence is that the stars are not seen in their true places, that is in the places which they truly held when the light left them (for what may have happened to them since we do not know at all—they may have gone out or exploded). Our earth is only moving slowly compared with the great haste of light : but still she is moving, and consequently there is "aberration"—a displacement due to the

ratio of the two velocities, easy enough to recognize now, but so difficult to apprehend for the first time that Bradley spent two years in worrying over the conundrum presented by his observations before he thought of the solution. It came to him unexpectedly, as often happens in such cases. In his own words—"at last when he despaired of being able to account for the phenomena which he had observed, a satisfactory explanation of them occurred to him all at once when he was not in search of it." He accompanied a pleasure party in a sail upon the river Thames. The boat in which they were was provided with a mast which had a vane at the top of it. It blew a moderate wind, and the party sailed up and down the river for a considerable time. Dr. Bradley remarked that every time the boat put about, the vane at the top of the boat's mast shifted a little, as if there had been a slight change in the direction of the wind. The sailors told him that this was due to the change in the boat, not the wind: and at once the solution of his problem was suggested. The earth running hither and thither round the sun resembles the boat sailing up and down the river: and the apparent changes of wind correspond to the apparent changes in direction of the light of a star. But now comes a point of detail—does the vane itself affect the wind just round it? And, similarly, does the earth itself by its movement affect the ether just round it, or the apparent direction of the light waves? This question suggested the famous Michelson and Morley experiment (*Phil. Mag.*, Dec. 1887). It is curious to think that in the little corner of the universe represented by the space available in a laboratory an experiment should be possible which alters our whole conceptions of what happens in the profoundest depths of space known to us, but

so it is. The laboratory experiment of Michelson and Morley was the first step in the great advance recently made. It discredited the existence of the virtual stream of ether which is the natural antithesis to the earth's actual motion. It was, indeed, open to question whether restrictions of a laboratory might not be responsible for the result: for the ether stream might exist, but the laboratory in which it was hoped to detect it might be in a sheltered eddy. When bodies move through the air, they encounter an apparent stream of opposing air, as all motorists know: but by using a glass screen shelter from the stream can be found. And even without such special screening, there may be shelter. When a pendulum is set swinging in ordinary air, it is found from experiments on clocks that it carries a certain amount of air along with it in its movement, although the portion carried probably clings closely to the surface of the pendulum. A very small insect placed in the region might be unable to detect the streaming of the air further out. In a similar way it seemed possible that as the earth moved through the ether such tiny insects as the physicists in their laboratories might be in a part of the ether carried along with the earth, in which they could not detect the streaming outside. But another laboratory experiment, this time by Sir Oliver Lodge, discredited this explanation, and it was then suggested as an alternative that distances were automatically altered by movement.

It may be well to explain briefly the significance of this alternative. The Michelson-Morley experiment depended on the difference between travelling up and down stream, and across it. To use a few figures may be the quickest way of making the point clear. Suppose a very wide, perfectly smooth stream

running at 3 miles an hour, and that oarsmen are to start from a *fixed* point O in midstream, row out in any direction to a distance of 4 miles from O, and back again to the starting-point O. Which is the best direction to choose? We shall probably all agree that it will be either directly up and down stream, or directly across it, and we may confine attention to these two directions. First suppose an oarsman A starts straight across stream. To keep straight he must set his boat at an angle to the stream. If he reaches his 4 mile limit in an hour, the stream has been virtually carrying him down 3 miles in a direction at right angles to his course: and the well-known relation between the sides of a right-angled triangle tells us that he has effectively pulled 5 miles in the hour. It will take him similarly an hour to come back, and the total journey will involve an effective pull of 10 miles.

Now suppose another oarsman, B, of equal skill elects to row up stream. In two hours he could pull 10 miles if there were no stream; but since meantime the stream has pulled him back 6 miles by "direct action" he will have only just reached the 4 mile limit from the start, and has still his return journey to go. No doubt he will accomplish this pretty quickly with the stream to help him, but his antagonist has already got home before he begins the return. We might have let him do his quick journey down stream first, but it is easy to see that this would gain him no ultimate advantage.

Michelson and Morley sent two rays of light on two journeys similar to those of the oarsmen A and B. The stream was the supposed stream of ether from east to west which should result from the earth's movement of rotation from west to east. They confidently expected the return of A before that of B, and were quite taken aback to find the

two reaching the goal together. In the aquatic analogy of which we have made use, it would no doubt be suspected that B was really the faster oar, which might be tested by interchanging the courses ; but there are no known differences in the velocity of light which would allow of a parallel explanation. There was, however, the possibility that the distances had been marked wrongly, and this was tested by interchanging them, without altering the "dead-heat."

Now there are several alternative explanations of this result. One is that the ether does not itself exist, and therefore there is no stream of it, actual or apparent ; and it is to this sweeping conclusion that modern reasoning, following recent experiments and observations, is tending. The possibility of saving the ether by endowing it with four dimensions instead of three is scarcely calculated to satisfy those who believed (until recently) that we knew more about the ether than about matter itself. They saved the ether for a time by an automatic shortening of all bodies in the direction of their movement, which explained the dead-heat puzzle. With the velocities used above, the goal attained by B must be automatically moved $\frac{4}{5}$ of a mile nearer the starting-point, so that B only rows $3\frac{1}{5}$ miles out and back instead of 4 miles. So gross a piece of cheating would enable B to make his dead-heat, but could scarcely escape detection. The shortening of the course required in the case of light is very minute indeed, because the velocities of the heavenly bodies are so small compared with that of light. If they could be multiplied a thousand times we might see some curious things, but we have no actual experience to guide a forecast.

It is a great triumph for Pure Mathematics that it should have devised a forecast for us in its own

peculiar way. Starting from axioms or postulates, Einstein, by sheer mathematical skill, making full use of the beautiful theoretical apparatus inherited from his predecessors, pointed ultimately to three observational tests, three things which must happen if the axioms and postulates were well founded. One of the tests—the movement of the perihelion of Mercury's orbit—had already been made and was awaiting explanation as a standing puzzle. Another—a displacement of lines in the spectrum of the sun—is still being made, the issue being not yet clear.

The third suggestion was that the rays of light from a star would be bent on passing near the sun by a particular amount, and this test has just provided a sensational triumph for Einstein. The application was particularly interesting because it was not known which of at least three results might be attained. If light were composed of material particles as Newton suggested, then in passing the sun they would suffer a natural deflection (the use of the adjective is an almost automatic consequence of modes of thought which we must now abandon) which we may call N . On Einstein's theory the deflection would be just twice this amount, $E = 2N$. But it was thought quite possible that the result might be neither N nor E but zero, and Professor Eddington remarked before setting out on the recent expedition that a zero result, however disappointing immediately, might ultimately turn out the most fruitful of all. That was less than a year ago. Perhaps a few dates are worth remembering. Einstein's theory was fully developed and stated in November, 1915, but news of it did not reach England (owing to the War) for some months. In 1917 the Astronomer Royal pointed out the special suitability of the Total Solar Eclipse of May, 1919,

as an occasion for testing Einstein's Theory. Preparations for two Expeditions were commenced—Mr. Hinks described the geographical conditions on the central line in November, 1917—but could not be fully in earnest until the Armistice of November, 1918. In November, 1919, the entirely satisfactory outcome was announced to the Royal Society and characterized by the President as necessitating a veritable revolution in scientific thought.

But when Mr. Brose brought me his translation of the pamphlet in the spring of 1919, the issue was still in doubt. He had become deeply interested in the new theory while interned in Germany as a civilian prisoner and had there made this translation. I encouraged him to publish it and opened negotiations to that end, but it was not until we enlisted the sympathy of Professor Eddington (on his return from the Expedition) and approached the Cambridge Press that a feasible plan of publication was found. Professor Eddington would have been a far more appropriate introducer; and it is only in deference to his own express wish that I have ventured to take up the pen that he would have used to much better purpose. One advantage I reap from the decision: I can express the thanks of Mr. Brose and myself to him for his practical help, and perhaps I may add those of a far wider circle for his own able expositions of an intricate theory, which have done so much to make it known in England.

H. H. TURNER

UNIVERSITY OBSERVATORY,
OXFORD.

November 30, 1919

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BIOGRAPHICAL NOTE

ALBERT EINSTEIN was born in March, 1879, in the town Ulm, situated on the banks of the Danube in Würtemberg, Germany. He attended school at Munich, where he remained till his sixteenth year.

His university studies extended over the period 1896-1900 at Zürich, Switzerland. He became a citizen of Zürich in 1901. During the following seven years he filled the post of engineer in the Patent Office, Bern. He accepted a call to Zürich as Professor Extraordinarius in 1910, which he, however, soon resigned in favour of a permanent chair in Prague University. In 1911 he decided to accept a similar post in Zürich. Since 1914 he has continued his researches in Berlin as a member of the Berlin Academy of Sciences.

His most important achievements are:

- 1905. The Special Theory of Relativity.
 - The discovery that all forms of energy possess inertia.
 - The law underlying the Brownian movement.
 - The Quantum-Law of the emission and absorption of light.
- 1907. The fundamental notions of the general theory of relativity.
- 1912. The recognition of the non-Euclidean nature of space-determination and its connection with gravitation.
- 1915. Gravitational field equations.
 - Explanation of the motion of Mercury's perihelion.

THE FOUNDATIONS OF EINSTEIN'S THEORY OF GRAVITATION

INTRODUCTION

TOWARDS the end of 1915 Albert Einstein brought to its conclusion a theory of gravitation on the basis of a general principle of relativity of *all* motions. His object was to create not a visual picture of the action of an attractive force between bodies, but rather a mechanics of the motions of the bodies relative to one another under the influence of inertia and gravity. To attain this difficult goal, it is true, many time-honoured views had to be sacrificed, but as a reward a standpoint was reached which had long seemed the highest aim of all who had occupied their minds with theoretical physics. The fact that these sacrifices are demanded by the new theory must, indeed, inspire confidence in it. For the unsuccessful attempts that have been made during the last centuries to fit the doctrine of gravitation satisfactorily into the scheme of natural science necessarily lead to the conclusion that this would not be possible without giving up many deeply-rooted ideas. As a matter of fact, Einstein reverted to the foundation pillars of mechanics as starting-points on which to build his theory, and he did not satisfy himself by merely reforming the Newtonian law in order to establish a link with the more recent views.

To get at an understanding of Einstein's ideas, we must compare the fundamental point of view adopted by Einstein with that of classical mechanics. We then recognize that a logical development leads from "the special" principle of relativity to the general theory, and simultaneously to a theory of gravitation.

§ I

THE "SPECIAL" THEORY OF RELATIVITY AS
A STEPPING STONE TO THE "GENERAL"
THEORY OF RELATIVITY

THE complete upheaval which we are witnessing in the world of physics at the present time received its impulse from obstacles which were encountered in the progress of electrodynamics. Yet the important point in the later development was that an escape from these difficulties was possible * only by founding mechanics on a new basis.

The development of electrodynamics took place essentially without being influenced by the results of mechanics, and without itself exerting any influence upon the latter, so long as its range of investigation remained confined to the electrodynamic phenomena of bodies *at rest*. Only after Maxwell's equations had furnished a foundation for these did it become possible to take up the study of the electrodynamic phenomena of *moving* media. All optical occurrences—and according to Maxwell's theory all these also belong to the sphere of electro-

* *Note*.—Most of the objections to the new development have, it is admitted, been raised because a branch of science which was not considered to have a just claim to deal with questions of mechanics, asserted the right of exercising a far-reaching influence upon the latter, extending even to its foundation. If, however, we trace these objections to their source, we discover that they are due to a wish to give mechanics the form of a purely *mathematical science*, similar to geometry, in spite of the fact that it is founded upon hypotheses which are essentially *physical*: up to the present, certainly, these hypotheses have not been recognized to be such.

dynamics—take place either between stellar bodies which are in motion relatively to one another, or upon the earth, which revolves about the sun with a velocity of about 30 kilometres per second, and performs, together with the sun, a translational motion of about the same order of magnitude through the region of the stellar system. Hence questions of great fundamental importance at once asserted themselves. Does the motion of a light-source leave its trace on the velocity of the light emitted by it? And what is the influence of the earth's motion on the optical phenomena which occur on its surface, for example, in optical experiments in a laboratory? An endeavour was therefore to be made to find a theory of these phenomena in which electrodynamic and mechanical effects occurred simultaneously (*vide* Note 1). Mechanics, which had long stood as a structure complete in every detail, had to stand the test as to whether it was capable of supplying the fitting arguments for a description of such phenomena. Thus the problem of electrodynamic events in the case of moving matter became at the same time a decisive problem of mechanics.

The first outstanding attempt to describe these phenomena for moving bodies was made by H. Hertz. He extended Maxwell's equations by additional terms so as also to express the influence of the motion of matter on electrodynamic phenomena, and in his extensions he adopted the view, characteristic for his theory, that the carrier of the electromagnetic field, the ether, everywhere participates in the motion of matter. Consequently, in his equations the state of motion of the ether, as denoting the state of the ether, occurs as well as the electromagnetic field. As is well known, Hertz's extensions cannot be brought into harmony with the results of observation, for example, that of Fizeau's experiment

(Note 2), so that they excite merely an historic interest as a land-mark on the road to an electrodynamics of moving matter. Lorentz was the first to derive from Maxwell's theory fundamental electrodynamic equations for moving matter which were in essential agreement with observation. He, indeed, succeeded in this only by renouncing a principle of fundamental importance, namely, by disallowing that Newton's and Galilei's principle of relativity of classical mechanics also holds for electrodynamics. The practical success of Lorentz's theory at first almost made us fail to see this sacrifice, but then the disintegration set in at this point which finally made the position of classical mechanics untenable. To understand this development we therefore require a detailed treatment of the principle of relativity in the fundamental equations of physics.

The principle of relativity of classical mechanics is understood to signify the consequence, which arises out of Newton's equations of motion, that two systems of co-ordinates, moving with uniform motion in a straight line with respect to one another, are to be regarded as fully equivalent for the description of events in the domain of mechanics. For our observations on the earth this means that any mechanical event on the surface of the earth—for example, the motion of a projected body—does not become modified by the circumstance that the earth is not at rest, but, as is approximately the case, is moving rectilinearly and uniformly. Yet this *postulate* of relativity does not fully characterize the Newtonian principle of relativity, even if it expresses that experimental fact which constitutes the essence of the principle of relativity. The postulate of relativity has yet to be supplemented by those formulæ of transformation by means of which the observer is able to transform the

co-ordinates x, y, z, t that occur in Newton's equations of motion into those of a system of reference which is moving uniformly and rectilinearly with respect to his own and which has the co-ordinates x', y', z', t' . Here the co-ordinates, x, y, z , that occur in the Newtonian equations denote throughout the results of measurement (obtained by means of rigid measuring rods according to the rules of Euclidean geometry), of the spatial positions of the bodies during the event in question, and the fourth co-ordinate t denotes the point of time assigned to the same event given by the position of the hands of a clock placed at the point at which the event occurs. Classical mechanics now supplemented the postulate of relativity above formulated by equations of transformation of the form :

$$x' = x - vt \quad y' = y \quad z' = z \quad t' = t$$

for the cases in which we are dealing with the co-ordinate relations of two systems of reference moving with the uniform velocity v in the direction of the x -axis with respect to each other. This group of so-called Galilei-transformations is distinguished, even in the case in which the direction of motion makes *any* angle with the co-ordinate axes, by the circumstance that the time-co-ordinate t always becomes transformed by the identity $t = t'$ into the time-values of the second system of reference; it is in this that the absolute character of the time-measures manifests itself in the classical theory. *Newton's equations of mechanics do not alter their form* if we substitute the co-ordinates x', y', z', t' in them for x, y, z, t by means of these equations of transformation. So long as we restrict ourselves to those systems of reference among all others that emerge out of each other as a result of transformations of the above type, there is no sense in talking

of absolute rest or absolute motion. For we may freely decide to regard either of two systems moving in such a way as the one that is at rest or in motion. According to classical mechanics it was, indeed, believed that only the Galilei-transformations could come into question when we were concerned with referring equivalent systems of reference to each other according to the principle of relativity. This, however, is not the case. The recognition of the fact that other equations of transformation may come into question for this purpose, and, indeed, may be chosen to suit the facts of observation which are to be accounted for, the recognition of this fact is the characteristic feature of the "special" theory of relativity of Lorentz-Einstein which replaced that of Galilei-Newton. Lorentz's fundamental equations of the electrodynamics of moving matter led to it. This system of electrodynamics, which is in satisfactory agreement with observation, is founded, in contradistinction to Hertz's theory, on the view of an absolutely rigid ether at rest. Its fundamental equations assume as its favoured system the co-ordinate system that is at rest in the ether.

These fundamental electrodynamical equations of Lorentz, however, change their form if, in them, we replace the co-ordinates x, y, z, t of a system of reference, initially chosen, by the co-ordinates x', y', z', t' of a system moving uniformly and rectilinearly with respect to the former by means of the transformation relationships. Must we infer from this that systems of reference which are moving uniformly and rectilinearly with respect to each other are *not* equivalent as regards electrodynamic events, and that there is no relativity principle of electrodynamics? No, this inference is not necessary, because, as remarked, the principle of relativity of classical mechanics with its group

of equations of transformation does not represent the *only* possible way of expressing the equivalence of systems of reference that are moving uniformly and rectilinearly with respect to each other. As we shall show in the sequel, the same postulate of relativity may be associated with another group of transformations. Nor did experiment seem to offer a reason for answering the above question in the affirmative. For all attempts to prove by optical experiments in our laboratories on the earth the progressive motion of the latter gave a negative result (Note 2). According to our observations of electrodynamic events in the laboratory the earth may be regarded equally well as at rest or in motion ; these two assumptions are *equivalent*.

This led to the definite conviction that in fact a principle of relativity holds for all phenomena, be their character mechanical or electrodynamic. But there can be only *one* such principle, and not one for mechanics and another for electrodynamics. For two such principles would annul each other's effects because we should be able to derive a favoured system from them in the case of events in which mechanical and electrodynamical events occur in conjunction, and this favoured system would allow us to talk with sense of absolute rest or motion with regard to it.

The one escape from this difficulty is that opened up by Einstein. In place of the relativity principle of Galilei and Newton we have to set another which comprehends the events of mechanics and electrodynamics. This may be done, without altering the *postulate* of relativity formulated above, by setting up a new group of transformations, which refer the co-ordinates of equivalent systems of reference to one another. The fundamental equations of mechanics must, certainly, then be remodelled so that

they preserve their form when subjected to such a transformation. Starting-points for this remodelling were already given. For it had been found empirically that Lorentz's fundamental equations of electrodynamics allowed new kinds of transformations of co-ordinates, namely, those of the form

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, y' = y, z' = z, t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where c = velocity of light *in vacuo*.

The new principle of relativity set up by Einstein is as follows: *Systems that are moving uniformly and rectilinearly with respect to each other are completely equivalent for the description of physical events. The equations of transformation that allow us to pass from the co-ordinates of one such system to those of another possible system, however, are not then* (for the case when both systems are moving parallel to their x -axes with the constant velocity v):—

$$x' = x - vt, y' = y, z' = z, t' = t$$

but

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, y' = y, z' = z, t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Thus the Galilei-Newton principle of relativity of classical mechanics and the Lorentz-Einstein "special" principle of relativity differ only in the form of the equations of transformation that effect the transition to equivalent systems of reference (Note 3).

Moreover, the relation of these two different

transformation formulæ to each other comes out clearly in the circumstance that the equations of transformation of Galilei and Newton may be derived by a simple passage to the limit from the new equations of Lorentz and Einstein. For if we assume the velocity v of each system with respect to the other to be very small compared with the velocity of light c , so that the quotient $\frac{v^2}{c^2}$ or $\frac{v}{c}$, respec-

tively, may be neglected in comparison with the remaining terms—an admissible assumption in all cases with which classical mechanics had so far dealt—the Lorentz-Einstein transformations pass over into those of Newton and Galilei.

It immediately suggests itself to us to ask what it is that compels us to give up the principle of relativity of classical mechanics, that is, what are the physical assumptions in its equations of transformation that stand, in contradiction with experience? The answer is that the principle of relativity of Newton and Galilei does not account for the facts of experience that emerge from Fizeau's and the Michelson-Morley experiment, and from which it may be inferred that the velocity of light has the particular character of a universal constant in the transformation relationships of the principle of relativity. In how far this peculiar property of the velocity of light receives expression in the new equations of transformation requires the following detailed explanation.

2 *The equations of transformation of the principle of relativity of Galilei and Newton contain a hypothesis* (which had hitherto not been recognized as such). For it had been tacitly assumed that the following assumption was fulfilled quite naturally: if an observer in a co-ordinate system S measure the

velocity v of the propagation of some effect or other, for example, a sound wave, then an observer in another co-ordinate system S' which is moving relatively to S , necessarily obtains a different measure for the velocity of propagation of the same action. This was to hold for *every finite* velocity v . Only infinite velocity was to be distinguished by the singular property that it was to come out in every system independently of its state of motion as having exactly the same value in all the measurements, namely, the value infinity.

This hypothesis—for we are here, of course, dealing only with a purely physical hypothesis—immediately suggested itself. Without further test there was no support for supposing that also a finite velocity, namely, the velocity of light, which the naïve point of view is inclined to endow with infinitely great velocity, would manifest the same singular property.

The fact, however, which the Michelson-Morley experiment helped us to become aware of was that the law of propagation for light is, for the observer, independent of any progressive motion of his system of reference, and has the property of isotropy (that is, equivalence of all systems) (*cf.* Note 2), so that it immediately suggests itself to us that the velocity of light is to be considered as having the same value for all systems of reference. The recognition of the fact thus arrived at was, without doubt, a surprise, but it will appear less strange to those who bear in mind the particular rôle of the velocity of light in the equations of Maxwell, the foundation of our theory of matter.

In consequence of this peculiarity, the velocity of light occurs in the equations of kinematics as a universal constant. To understand this better we pursue the following argument. Long before the

advent of the questions of electrodynamic phenomena in moving bodies we might, on grounds of principle, have suggested quite generally the question: how are the co-ordinates in two systems of reference that are moving uniformly and rectilinearly with respect to each other to be referred to each other? We should have been able to attack the purely *mathematical* problem with a full consciousness of the assumptions contained in the hypotheses, as was actually done later by Frank and Rothe (Note 4). We then arrive at equations of transformation that are much more *general* than those written down on p. 9. By taking into account the special conditions that nature manifests to us, for example the isotropy of space, we may derive from them particular forms, the hypothetical assumptions contained in which come clearly to view. Now, in these general equations of transformation a quantity occurs that deserves special notice. There are "invariants" of these equations of transformation, that is, quantities that preserve their value even when such a transformation is carried out. Among these invariants there is a velocity. This signifies the following: if an effect propagates itself in one system with the velocity v , then in general the velocity of propagation of the same effect in another system is other than v , if the second system is moving relatively to the first. Only the invariant velocity preserves its value in all systems, no matter with what velocity they be moving relatively to one another. The value of this invariant velocity enters as a characteristic constant into the equations of transformation. Hence, if we wish to find those transformation relations that hold *physically*, we must find out the singular velocity that plays this fundamental part. To determine it is the task of the experimental

physicist. If he sets up the *hypothesis* that a finite velocity can never be such an invariant, the *general* equations of transformation degenerate into the transformation-relationships of the principle of relativity of Galilei and Newton. (This hypothesis was made, albeit unconsciously, in Newtonian mechanics.) It had to be discarded after the results of the Michelson-Morley and Fizeau's experiment had justified the view that the *velocity of light* c plays the part of an invariant velocity. Then the *general equations* of transformation degenerate into those of the "special" principle of relativity of Lorentz and Einstein.

This remodelling of the co-ordinate-transformations of the principle of relativity led to discoveries of fundamental importance, as, for example, to the surprising fact that the conception of the "simultaneity" of events at different points of space, the conception on which all time-measurements are based, has only a relative meaning, that is, that two events that are simultaneous for one observer will not, in general, be simultaneous for another.* This deprived time-values of the

* The assertion, "At a particular point of the earth the sun rises at 5 o'clock 10' 6"," denotes that "the rising of the sun at a particular point of the earth is simultaneous with the arrival of the hands of the clock at the position 5 o'clock 10' 6" at that point of the earth." In short, the determination of the point of time for the occurrence of an event is the determination of the *simultaneity* of happening of *two* events, of which one is the arrival of the hands of a clock at a definite position at the point of observation. The comparison of the points of time at which one and the same event occurs, as noted by several observers situated at different points, requires a convention concerning the times noted at the different points. The analysis of the necessary conventions led Einstein to the fundamental discovery that the conception "simultaneous" is only relative inasmuch as the relation of time-measurements to one another in systems that are moving relatively to one another is dependent on their state of motion. This was the starting-point for the arguments that led to the enunciation of the "special principle of relativity."

absolute character which had previously been a great point of distinction between them and space co-ordinates. So much has been written in recent years about this question that we need not treat it in detail here.

The new form of the equations of transformation by no means exhausts the whole effect of the principle of relativity upon classical mechanics. The change which it brought about in the conception of mass was almost still more marked.

Newtonian mechanics attributes to every body a certain inertial mass, as a property that is in no wise influenced by the physical conditions to which the body is subject. Consequently, the Principle of the Conservation of Mass also appears in classical mechanics as independent from the Principle of the Conservation of Energy. The special principle of relativity shed an entirely new light on these circumstances when it led to the discovery that energy also manifests inertial mass, and it hereby fused together the two laws of conservation, that of mass and that of energy, to a single principle. The following circumstance moves us to adopt this new view of the conception of mass.

The equations of motion of Newtonian mechanics do not preserve their form when new co-ordinates have been introduced with the help of the Lorentz-Einstein transformations. Consequently, the fundamental equation of mechanics had to be remodelled accordingly. It was then found that Newton's Second Law of Motion : *force* = *mass* $\times$ *accel.* cannot be retained, and that the expression for the kinetic energy of a body may no longer be furnished by the simple expression $\frac{1}{2}mv^2$, which involves the mass and the velocity. Both these results are consequences of the change which we found necessary to make in our view of the nature of the mass

of matter. The new principle of relativity and the equations of electrodynamics led, rather, to the fundamentally new discovery that inertial mass is a property of every kind of energy, and that a point-mass, in emitting or absorbing energy, decreases or increases, respectively, in inertial mass, as is shown in Note 5 for a simple case. The new kinematics thereby disposes of the simple relation between the kinetic energy of a body and its velocity relatively to the system of reference. The simplicity of the expression for the kinetic energy in Newtonian mechanics rendered possible the revolution of the energy of a body into that (kinetic) of its motion and of the internal energy of the body, which is independent of the former. Let us consider, for example, a vessel containing material particles, no matter of what kind, in motion. If we resolve the velocity of each particle into two components, namely, into the velocity, common to all, of the centre of gravity and the accidental velocity of a particle relative to the centre of gravity of the system, then, according to the formulæ of classical mechanics, the kinetic energy divides up into two parts: one that contains exclusively the velocity of the centre of gravity and that represents the usual expression for the kinetic energy of the whole system (mass of the vessel *plus* the mass of the particles), and a second component that involves only the inner velocities of the system. This category of internal energy is no longer possible so long as the expression for the kinetic energy contains the velocity not merely as a quadratic factor; so we are led to the view that the internal energy of the body comes into expression in the energy due to its progressive motion, and, indeed, as an increase in the inertial mass of the body.

This discovery of the inertia of energy created

an entirely new starting-point for erecting the structure of mechanics. Classical mechanics regards the inertial mass of a body as an absolute, invariable, characteristic quantity. The special theory of relativity, it is true, makes no direct mention of the inertial mass associated with *matter*, but it tells us that *every* kind of *energy* has also inertia. But, as every kind of matter has at all times a probably enormous amount of latent energy, its inertia is composed of two components; the inertia of the matter and the inertia of its contained energy, which consequently alters with the amount of the energy-content. This view leads us naturally to ascribe the phenomenon of inertia in bodies to their energy-content altogether.

Thus, there arose the important task of absorbing these new discoveries concerning the nature of inert mass into the principles of mechanics. A difficulty hereby arose which, in a certain sense, pointed out the limits of achievement of the special theory of relativity. One of the fundamental facts of mechanics is the equality of the inertial and gravitational mass of a body. It is on the supposition that this is true that we determine the mass of a body by measuring its weight. The *weight* of a body is, however, only defined *with reference to a gravitational field* (Note 18): in our case, with reference to the earth. The idea of *inertial mass* of a body is, however, introduced as an attribute of matter *without any reference whatsoever to physical conditions external to the body*. How does the mysterious coincidence in the values of the inertial and gravitational mass of a body come about?

Nor does the special theory of relativity provide an answer to this question. The special theory of relativity does not even preserve the equality in

the values of inertia and gravitational mass ; a fact which is to be reckoned amongst the most firmly established facts in the whole of physics. For, although the special theory of relativity makes allowance for an *inertia* of energy, it makes none for a *gravitation* of energy. Consequently, a body which absorbs energy in any way will register a gain of inertia but not of weight, thereby transgressing the principle of the equality of inertial and gravitational mass ; for this purpose a theory of gravitational phenomena, a theory of gravitation, is required. The special theory of relativity can, therefore, be regarded only as a *stepping-stone* to a more general principle, which orders gravitational phenomena satisfactorily into the principles of mechanics.

This is the point where Einstein's researches towards establishing a general theory of relativity set in. He has discovered that, by extending the application of the relativity-principle to accelerated motions, and by introducing gravitational phenomena into the consideration of the fundamental principles of mechanics, a new foundation for mechanics is made possible, in which all the difficulties occurring up to the present are solved. Although this theory represents a consistent development of the knowledge gathered by means of the special theory of relativity, it is so deeply rooted in the substructure of our principles of knowing, in their application to physical phenomena, that it is possible thoroughly to grasp the new theory only by clearly understanding its attitude toward these guiding lines provided by the theory of knowledge.

I shall, therefore, commence the account of his theory by discussing two general postulates, which should be fulfilled by every physical law, but neither

of which is satisfied in classical mechanics : whereas their strict fulfilment is a characteristic feature of the new theory. Here we have thus a suitable point of entry into the essential outlines of the general theory of relativity.

§ 2

TWO FUNDAMENTAL POSTULATES IN THE
MATHEMATICAL FORMULATION OF PHYSICAL LAWS

NEWTON had established the simple and fruitful law that two bodies, even when they are not visibly connected with one another, as in the case of the heavenly bodies, exert a mutual influence, attracting one another with a force directly proportional to the product of their masses, and inversely proportional to the square of the distance between them. But Huygens and Leibniz refused to acknowledge the validity of this law, on the ground that it did not satisfy a fundamental condition to which every physical law is subject, viz. that of *continuity* (continuity in the transmission of force, action "by contact" in contradistinction to action "at a distance"). How were two bodies to exert an influence upon one another without a medium between them to transmit the action? The demand for a satisfactory answer to this question became, in fact, so imperative that finally, in order to satisfy it, the existence of a substance which pervaded the whole of cosmic space and permeated all matter—the "luminiferous ether"—was assumed, although this substance seemed to be condemned to remain intangible and invisible (i.e. imperceptible to the senses for all time) and had to be endowed with all sorts of contradictory properties. In the course of time, however,

there arose in opposition to such assumptions the more and more definite demand that, *in the formulation of physical laws, only those things were to be regarded as being in causal connection which were capable of being actually observed*: a demand which doubtless originates from the same instinct in the search for knowledge as that of continuity, and which really gives the law of causality the true character of an *empirical* law, i.e. one of actual *experience*.

The consistent fulfilment of these two postulates combined together is, I believe, the mainspring of Einstein's method of investigation; this imbues his results with their far-reaching importance in the construction of a physical picture of the world. In this respect his endeavours will probably not encounter any opposition in the matter of principle on the part of scientists. *For both postulates—(1) that of continuity and (2) that of causal relationship between only such things as lie within the realm of observation—are of an inherent nature, i.e. contained in the very nature of the problem. The only question that might be raised is whether it is expedient to abandon such useful working hypotheses as "forces at a distance."*

The principle of continuity requires that all physical laws allow of formulation as differential laws, i.e. physical laws must be expressible in a form such that the physical state at any point is completely determined by that of the point in its immediate neighbourhood. Consequently, the distances between points, which are at *finite* distances from one another, must not occur in these laws, but only those between points infinitely near to one another. The law of attraction of Newton given above, inasmuch as it involves "action at a distance," disobeys the first postulate.

The second postulate, that of a stricter form of expression for causality in its occurrence in physical laws, is intimately connected with a general theory of relativity of motions. Such a general principle of relativity requires that all possible systems of reference in nature be equivalent for the description of physical phenomena, and hence it avoids the introduction of the very questionable conception of absolute space which, for reasons we know (see § 4), could not be circumvented by Newtonian mechanics. A general theory of relativity would, in excluding the fictitious quantity "absolute space," reduce the laws of mechanics to motions of bodies relative to one another, which are actually and exclusively what we observe. Thus, its laws would be founded on observed facts more completely than are those of classical mechanics.

The rigorous application of the principles of continuity and relativity in their general form penetrates deeply into the problem of the mathematical formulation of physical laws. It will, therefore, be essential at the outset to enter into a consideration of the principles involved in the latter process.

§ 3

CONCERNING THE FULFILMENT OF THE TWO POSTULATES

A PHYSICAL law is clothed in mathematical language by setting up a formula. This comprises, and represents in the form of an equation, all measurements which numerically describe the event in question. We make use of such formulæ, not only in cases in which we have the means of checking the results of our calculations at any moment actually at our disposal, but also when the corresponding measurements cannot really be carried out *in practice*, but have to be *imagined*, i.e. only take place in our minds: e.g. when we speak of the distance of the moon from the earth, and express it in metres, as if it were really possible to measure it by applying a metre-rule end to end.

By means of this expedient of analysis we have extended the range of exact scientific research far beyond the limits of measurement actually accessible in practice, both in the matter of immeasurably large, as well as in that of immeasurably small, quantities. Now, when such a formula is used to describe an event, symbols occur in it that stand for those quantities which are, in a certain sense, the ground elements of the measurements, with the help of which we endeavour to grip the event; thus, for example, in the case of all spatial measurements, symbols for the "length" of a rod,

the "volume" of a cube, and so forth. In creating these ground elements of spatial elements we had hitherto been led by the idea of a rigid body which was to be freely movable in space without altering any of its dimensional relationships. By the repeated application of a rigid unit measure along the body to be measured we obtained information about its dimensional relationships. This idea of the ideal rigid measuring rod, which is only partially realizable in practice, on account of all sorts of disturbing influences such as the expansion due to heat, represents the fundamental conception of the geometry of measure.

The discovery of suitable mathematical terms, which can be inserted in a formula as symbols for definite physical magnitudes of measurements, such as e.g. *length* of a rod, *volume* of a cube, etc., in order to shift the responsibility, as it were, for all further deductions upon analysis, is one of the fundamental problems of theoretical physics and *is intimately connected with the two postulates* enunciated in § 2.

To realize this fully, we must revert to the foundations of geometry, and analyse them from the point of view adopted by Helmholtz in various essays, and by Riemann in his inaugural dissertation of 1854: "On the hypotheses which lie at the bases of geometry." Riemann points almost prophetically to the path now taken by Einstein.

(a) THE LINE-ELEMENT IN THE THREE-DIMENSIONAL MANIFOLD OF POINTS IN SPACE, EXPRESSED IN A FORM COMPATIBLE WITH THE TWO POSTULATES

Every point in space can be singly and unambiguously defined by the three numbers x_1, x_2, x_3 ,

which may be regarded as the co-ordinates of a rectangular system of co-ordinates, and which distinguish it from all other points; a continuous variation of these three numbers enables us to specify every single point of space in turn. The assemblage of points in space represents, in Riemann's notation, "a multiply extended magnitude" (an n -fold manifoldness or manifold) between the single elements (points) of which a *continuous* transition is possible. We are familiar with diverse *continuous* manifolds, e.g. the system of colours, of tones and various others. A feature which is common to all of them is that, in order to specify a single element out of the entire manifold (to define a *particular* point, a *particular* colour, or a *particular* tone), a characteristic number of magnitude-determinations, i.e. co-ordinates, is required: this characteristic number is called the *dimensions* of the respective manifold. Its value is three for space, two for a plane, one for a line. The system of colours is a continuous manifold of the dimension three, corresponding to the three "primary" colours, red, green, and violet, by mixing which in due proportions *every* colour can be produced.

But the assumption of continuity for the transition from one element to another in the same manifold, and the determination of the dimensions of the latter, does not give us any information about the possibility of comparing limited parts of the same manifold with one another, e.g. about the possibility of comparing two tones with one another or two single colours; i.e. nothing has yet been stated about the metric relations (measure-conditions) of the manifold, about the nature of the scale, according to which measurements can be undertaken within the manifold. In order to be able to do this, we must allow experience to give us the facts

from which to establish the metric (measure-) laws which hold for each particular manifold (space-points, colours, tones) under various physical conditions; these metric laws will be different according to the set of empirical facts chosen for this purpose.*

In the case of the manifold of space-points, experience has taught us that *finite rigid* point-systems can be freely moved in space without altering their form or dimensions; the conception of "congruence" which has been derived from this fact, has become a vital factor for a measure-determination.† It sets us the problem of building up a mathematical expression from the numbers x_1, x_2, x_3 , and y_1, y_2, y_3 , which are assigned to two definite points in space, and which we may imagine as the end-points of a rigid measuring rod, such that this expression may be regarded as a measure of the distance between them, that is, as an expression for the length of the rod, and may be introduced as such into the formulæ expressing physical laws.

The equations of physical laws, which—in order to fulfil the conditions of continuity—must be differential laws, contain only the distances ds , of infinitely near points, so-called *line-elements*. We must, therefore, inquire whether our two postulates of § 2 have any influence upon the analytical expression for the *line-element* ds , and, if so, which expression for the latter is compatible with both. Riemann demands of a line-element in the first place that it can be *compared* in respect to its *length* with every other line-element independently of its position and direction. This is a distinguishing characteristic of the metric conditions ("measure relations") prevalent in space; in practice it

* *Vide* Note 2.

† *Vide* Note 3.

denotes that the rods must be freely movable. This peculiarity does not exist, for instance, in the manifold of tones or in that of colours (*vide* Note 7). Riemann formulates this condition in the words, "that lines must have a length independent of their position and that every line is to be measurable by means of any other." He then discovers that: if x_1, x_2, x_3 and $x_1 + dx_1, x_2 + dx_2, x_3 + dx_3$ respectively denote two infinitely near points in space and if the continuously variable numbers x_1, x_2, x_3 are *any* co-ordinates whatsoever (not e.g. necessarily rectilinear), then the square root of an always positive, integral, homogeneous function of the second degree in the differentials dx_1, dx_2, dx_3 has all the properties* which the line-element, being the expression for the length of an infinitely small rigid measuring rod, must exhibit. We thus find that

$$ds = \sqrt{g_{11}dx_1^2 + g_{12}dx_1dx_2 + \dots + g_{33}dx_3^2},$$

in which the coefficients $g_{\mu\nu}$ are continuous functions of the three variables x_1, x_2, x_3 , gives us an expression for the line-element at the point x_1, x_2, x_3 .

In this expression no assumptions are made concerning the nature of the co-ordinates that are represented by the three variables, x_1, x_2, x_3 , that is, concerning particular metrical properties of the manifold that go beyond the postulate of the freedom of movement of the measuring rods. But, if we demand, in particular, that each point of the manifold may be fixed by means of rectangular Cartesian co-ordinates, whereby particular assumptions are made concerning the possible ways of placing the measuring rods, then the line-element,

* *Vide* Note 8.

expressed in these special variables, assumes the form

$$ds = \sqrt{dx^2 + dy^2 + dz^2}.$$

Hitherto this expression has always been introduced for the length of the line-element in all physical laws. It is contained in the more general expression of Riemann's line-element ds as the special case

$$g_{\mu\nu} \begin{cases} = 1, & \mu = \nu \\ = 0, & \mu \neq \nu. \end{cases}$$

By restricting ourselves to this special form of the line-element we are enabled to use the measure laws of Euclidean geometry in all our space-measurements.

But this particular assumption concerning the metrical constitution of space contains the hypothesis, as Helmholtz has shown in a detailed discussion, that *finite* rigid point-systems, i.e. *finite* fixed distances, are capable of unrestrained motion in space, and can be made (by superposition) to coincide with other (congruent) point-systems. With respect to the postulate of *continuity*, this hypothesis seems inconsistent, in so far as it introduces implicit statements about *finite* distances into purely differential laws, in which only *line-elements* occur; but it does not *contradict* the postulate.

The postulate of the relativity of all motion adopts a different attitude towards the possibility of giving the line-element the Euclidean form in particular.*

* Strictly speaking, I should at this juncture state in anticipation that the above investigations can manifestly also be so generalized as to be valid for the four-dimensional *space-time* manifold, in which all events actually take place, and that the transformation-formulæ apply to the four variables of this manifold. In these general remarks the neglect of the fourth dimension is of no importance. This statement will be justified later in § 3 (b).

According to the principle of the relativity of all motions, all systems, which come about owing to relative motions of bodies towards one another, may be regarded as fully equivalent. The laws of physics must, therefore, preserve their form in passing from one such system to another; i.e. the transformation-formulæ of the variables x_1, x_2, x_3 which perform this transition to another system, must not alter the analytical expression for the physical law under consideration.

This leads us to set up a principle of relativity which will be called the general principle of relativity in the sequel. It demands the invariance of physical laws with respect to arbitrary continuous substitutions of the four variables. Moreover, the line-element that occurs in it must preserve its form when subjected to any arbitrary transformations whatsoever. This condition is fully satisfied by the line-element

$$ds = \sqrt{g_{11}dx_1^2 + g_{12}dx_1dx_2 + \dots + g_{33}dx_3^2},$$

in which no restrictive reservations of any description are made as to what the co-ordinates x_1, x_2, x_3 are to signify. The Euclidean line-element

$$ds = \sqrt{dx^2 + dy^2 + dz^2},$$

on the other hand, preserves its form *only* for transformations of the *special* theory of relativity, which confine themselves to systems moving *uniformly* and *rectilinearly*. Consequently, the element of arc must be adapted to the further requirements of a general theory of relativity so that it preserves its form after any substitutions whatsoever.

The choice of the expression $ds^2 = \sum_1^3 g_{\mu\nu} dx_\mu dx_\nu$ to

represent the line-element in physical laws is, in spite of its very general character, still to be regarded as a hypothesis, as Riemann has already pointed out. For there are other functions of the differentials dx_1, dx_2, dx_3 —such as e.g. the fourth root of a homogeneous differential expression of the fourth degree in these variables—which could provide a measure for the length of the line-element (*vide* Note 9). But at present there is no ground for abandoning the simplest general expression for the line-element (*viz.* that of the second degree), and adopting more complicated functions. Within the range (of fulfilment) of the two postulates, which we have imposed upon every description of physical events, the former expression for ds satisfies all requirements. Nevertheless, it must never be forgotten that the choice of an analytical expression for the line-element always contains a hypothetical factor; and it is the duty of the physicist to remain fully conscious of this fact at all times, without being in any way prejudiced. It is for this reason that Riemann closes his essay with the following remarks, which impress one particularly with their great importance for the present time:*

“The question of the validity of the hypotheses of geometry in the infinitely small is bound up with the question of the ground of the metric relations of space. In this question, which we may still regard as belonging to the doctrine of space, is found the application of the remark made above; that in a discrete† manifold, the principle or character of its metric relations is already given in the notion of the manifold, whereas in a

* B. Riemann, *Über die Hypothesen, welche der Geometrie zugrunde liegen*. New edn., annotated by H. Weyl, Berlin: Springer & Co., 1919.

† *Vide* Note 6.

continuous manifold this ground has to be found elsewhere, i.e. has to come from outside. Either, therefore, the reality which underlies space must form a discrete * manifold, or we must seek the ground of its metric relations (measure-conditions) outside it, in binding forces which act upon it.

A decisive answer to these questions can be obtained only by starting from the conception of phenomena which has hitherto been justified by experience, and of which Newton laid the foundation, and then making in this conception the successive changes required by facts which admit of no explanation on the old theory; researches of this kind, which commence with general notions, cannot be other than useful in preventing the work from being hampered by too narrow views, and in keeping progress in the knowledge of the inter-connections of things from being checked by traditional prejudices.

This carries us over into the sphere of another science, that of physics, into which the character and purpose of the present discussion will not allow us to enter."

That is to say: according to Riemann's view these questions are to be solved by starting from Newton's view of physical phenomena, and compelled by facts which do not allow of any explanation by it, gradually remoulding it. This is what Einstein has done. The "binding forces," to which Riemann points, will be found again in Einstein's theory. As we shall see in the fifth chapter, Einstein's theory of gravitation is based upon the view that the gravitational forces are the "binding forces," i.e. they represent the "inner ground" of the metric conditions (measure-relations) in space.

* *Vide* Note 10.

(b) THE LINE-ELEMENT IN THE FOUR-DIMENSIONAL
MANIFOLD OF SPACE-TIME POINTS, EXPRESSED
IN A FORM COMPATIBLE WITH THE TWO POST-
ULATES

The measure-conditions, which we were to take as a basis for the formulation of physical laws, could have been treated immediately in connection with the four-dimensional manifold of space-time points. For the special theory of relativity has led us to make the important discovery that the space-time-manifold has uniform measure-relations in its four dimensions. Nevertheless, I wish to treat time-measurements separately ; for one reason that it is just this result of the relativity-theory which has experienced the greatest opposition at the hands of supporters of classical mechanics ; and for another that classical mechanics is also obliged to establish certain conditions about time-measurement, but that it never succeeded in establishing agreement on this point. The difficulties with which classical mechanics had to contend are contained in its fundamental conceptions. The law of inertia, particularly, was a permanent factor of discord that caused the foundations of mechanics to be incessantly criticised. And since the foundations of time-measurement had been brought into close relationship with the law of inertia, these critical attacks applied to them likewise.

In Galilei's law of inertia, a body which is not subject to external influences continues to move with uniform motion in a straight line. Two determining elements are lacking, viz. the reference of the motion to a definite system of co-ordinates, and a definite time-measure. Without a time-measure one cannot speak of a *uniform* velocity.

Following a suggestion by C. Neumann,* the law of inertia has itself been adduced to give a definition of a time-measure in the form: "Two material points, both left to themselves, move in such a way that equal lengths of path of the one correspond to equal lengths of path of the other." On this principle, into which time-measure does not enter explicitly, we can define "equal intervals of time as such, within which a point, when left to itself, traverses equal lengths of path."

This is the attitude which was also taken up by L. Lange, H. Seeliger, and others, in later researches. Maxwell selected this definition too (in "Matter and Motion"). On the other hand, H. Streintz † (following Poisson and d'Alembert) has demanded the disconnection and independence of the time-measure from the law of inertia, on the ground that the roots of the time-concept have a deeper and more general foundation than the law of inertia. According to his opinion, *every* physical event, which can be made to take place again under exactly the same conditions, can serve for the determination of a time-measure, inasmuch as every *identical* event must claim precisely the same duration of time; otherwise, an ordered description of physical events would be out of the question. In point of fact, the clock is constructed on this principle. It is this principle which enables an observer to undertake a time-measurement *at least for his place of observation*. The reduction of time-measurements to a dependence upon the law of inertia, on the other hand, leads to an unobjectionable *definition of equal lengths of time*; but the *measurement* of the equal paths traversed by uniformly moving bodies, and the *establishment* of a unit of time involved therein, are only then possible for a place of observation,

* *Vide* Note 11.

† *Vide* Note 12.

when the observer and the moving body are in constant connection, e.g. by light-signals. It cannot, however, be straightway assumed that two observers, who are in rectilinear motion relatively to one another, and, therefore, according to the law of inertia, equivalent as reference systems, would in this manner gain identical results in their time-measurements. Poisson's idea thus leads to a satisfactory time-measurement *for a given place of observation* itself; i.e. in a certain sense it allows the construction of a clock for that place. But it does not broach the question of the time-relations of *different* places with *one another* at all; whereas Neumann's suggestion leads directly to those questions which have been a centre of discussion since Einstein's enunciation of the relativity-principle.

In the endeavour to reduce classical mechanics to as small a number of principles as possible, in perfect agreement with one another, writers resorted to *ideal-constructions* and *imaginary experiments*.

Yet no one conceived the idea that in fixing a unit of time on the basis of the law of inertia, that is, by measuring a length (the distance traversed), the state of motion of the observer might exert an influence. It was assumed that the data obtained from the necessary observations had an absolute meaning quite independent of the conditions of observation when simultaneous moments were chosen and a length was evaluated. As Einstein has shown, however, this is not the case. Rather, this recognition of the relativity of space- and time-measurements formed the starting point of his principle of relativity (Note 13). It is a necessary consequence of the universal significance of the velocity of light, of which we spoke in the first section. Its recognition furnished us at once with the correct formulæ of transformation, allowing

us to relate the space-time measurements of systems moving uniformly and rectilinearly with respect to each other, and this is what we are concerned with in Neumann's suggestion of fixing a measure of time with the aid of the law of inertia. In the new equations of transformation, t' is not identically equal to t , but rather

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The time-measurements in the second system which is moving relatively to the first are thus essentially conditioned by the velocity v of each relative to the other. Consequently, the fixing of a measure of time on the basis of the law of inertia, as proposed by Neumann, does not at all lead to the result that the time-measurements are entirely independent of the state of motion of the systems with respect to each other, as assumed in classical mechanics. Only when the researches of Einstein concerning the special theory of relativity had been carried out, did the fundamental assumptions of our time-measurements become fully cleared up, and thus a serious shortcoming in classical mechanics was made good.

That such a fundamental revision of the assumptions made regarding time-measurements became necessary only after so great a lapse of time, is to be explained by the fact that even the velocities which occur in astronomy are so small, in comparison with the velocity of light, that no serious discrepancies could arise between theory and observation. So it occurred that the weaknesses of the theory—in particular, those due to the motional relations

of various systems to one another—did not come to light until the study of electronic motions, in which velocities of the order of that of light occur, proved the insufficiency of the existing theory.

The details of the effects, which result from the relativity of space-time measurements, have so frequently been discussed in recent years that it is only possible to repeat what has already often been said. The essential point in the discussion of this section is the recognition of the fact that space and time represent a *homogeneous* manifold of “four” dimensions, with *homogeneous measure-relations* (*vide* Note 14). Consequently, to be consistent, we must apply the arguments of the preceding § 3 (a) about the measure-relations to the four-dimensional space-time-manifold; and, in view of the two fundamental postulates (1) of continuity and (2) of relativity, and including the time-measurement as the fourth dimension, we must select for our line-element the expression:

$$ds^2 = g_{11}dx_1^2 + g_{12}dx_1dx_2 + \dots + g_{34}dx_3dx_4 + g_{44}dx_4^2,$$

in which the $g_{\mu\nu}$ ($\mu\nu = 1, 2, 3, 4$) are functions of the variables x_1, x_2, x_3, x_4 .

Hitherto we have been led to adopt this much more general attitude towards the questions of the metric laws involved in physical formulæ merely by the desire not to introduce, from the very outset, more assumptions into the formulations of physical laws than are compatible with both postulates, and to bring about a deeper appreciation of the points of view, to which the *special* theory of relativity has led us.

We can briefly summarize by saying: the adoption of *Euclidean* metric-conditions (measure-relations) is *compatible* with the postulate of *continuity*; though the special assumptions thereby involved

appear as restrictive or limiting hypotheses, which need not be made. But the second postulate, the reduction of *all* motions to relative motions, *compels* us to abandon the Euclidean measure-determination (cf. p. 43). A description of the difficulties still remaining in mechanics will make this step clear.

§ 4

THE DIFFICULTIES IN THE PRINCIPLES OF
CLASSICAL MECHANICS

THE foundations of classical mechanics cannot be exhaustively described in a narrow space. I can only bring the unfavourable side of the theory into prominent view for the present purpose, without being able to do justice to its great achievements in the past. All doubts about classical mechanics set in at the very commencement with the formulation of the law which Newton places at its head, the formulation of the law of inertia.

As has already been emphasized on page 31, the assertion that a point-mass which is left to itself moves with uniform velocity in a straight line, omits all reference to a definite co-ordinate system. An insurmountable difficulty here arises: Nature gives us actually no co-ordinate system, with reference to which a uniform rectilinear motion would be possible. For as soon as we connect a co-ordinate system with any body such as the earth, sun, or any other body—and this alone gives it a physical meaning—the first condition of the law of inertia (viz. freedom from external influences) is no longer fulfilled, on account of the mutual gravitational effects of the bodies. One must accordingly either assign to the motion of the body a meaning in itself, i.e. grant the existence of motions relative

to "absolute" space, or have recourse to mental experiments by following the example of C. Neumann and introducing a hypothetical body Alpha, relative to which a system of axes is defined, and with reference to which the law of inertia is to hold (Inertial system, *vide* Note 15). The alternatives with which one is faced are highly unsatisfactory. The introduction of absolute space gives rise to the oft-discussed conceptual difficulties which have gnawed at the foundations of Newton's mechanics. The introduction of the system of reference Alpha certainly takes the relativity of motions so far into account, that all systems in uniform motion relative to an Alpha-system are established as *equivalent* from the very outset, but we can affirm with certainty that there is no such thing as a visible Alpha-system, and that we shall never succeed in arriving at a final determination of such a system. (It will, at most, be possible, by progressively taking account of the influences of constellations upon the solar system and upon one another, to approximate to a system of coordinates, which could play the part of such an inertial system *with a sufficient degree of accuracy*.) As a result of this objection, the founder of the view himself, C. Neumann, admits that it will always be somewhat unsatisfactory and enigmatical, and that mechanics, based on this principle, would indeed be a very peculiar theory.

It therefore seems quite natural that E. Mach (*vide* Note 16) should be led to propose that the law of inertia be so formulated that its relations to the stellar bodies are directly apparent. "Instead of saying that the direction and speed of a mass u remains constant in space, we can make use of the expression that the mean acceleration of the mass μ relative to the masses $m, m', m'' \dots$ at

distances $r, r', r'' \dots$, respectively, is zero or

$$\frac{d^2}{dt^2} \cdot \frac{\sum mr}{\sum m} = 0.$$

The latter expression is equivalent to the former statement, as soon as a sufficient number and sufficiently great and extensive masses are taken into consideration. . . .” This formulation cannot satisfy us. For, in addition to a certain requisite accuracy, the character of a “contact” law is lacking, so that its promotion to the rank of a fundamental law (in place of the law of inertia) is quite out of the question.

The inner ground of these difficulties is without doubt to be found in an insufficient connection between fundamental principles and observation. As a matter of actual fact, we only observe the motions of bodies relatively to one another, and these are never absolutely rectilinear nor uniform. Pure inertial motion is thus a conception deduced by abstraction from a mental experiment—a mere fiction.

However necessary and fruitful a mental experiment may often be, there is the ever-present danger that an abstraction which has been carried unduly far loses sight of the physical contents of its underlying notions. And so it is in this case. If there is no meaning for our understanding in talking of the “motion of a body” in space, as long as there is only this *one* body present, is there any meaning in granting the body attributes such as *inertial mass*, which arise only from our observation of *several* bodies, moving *relatively* to one another? If not, then we cannot attach to the conception “inertial mass of a body,” an absolute significance, that is, a meaning which is independent of all other physical conditions, as has hitherto been done.

Such doubts received fresh strength when the special theory of relativity endowed *every* form of energy with inertia (*vide* Note 17).

The results of the special theory of relativity entirely unhinged our view of the inertia of matter, for they robbed the theorem concerning the equality of inertial and gravitational mass of its strict validity. A body was now to have an *inertial* mass varying with its contained internal energy, without its *gravitational* mass being altered. But the mass of a body had always been ascertained from its weight, without any inconsistencies manifesting themselves (*vide* Note 18).

A difficulty of such a fundamental character could come to light only owing to the theorem of the equality of inertial and gravitational mass not being sufficiently interwoven with the underlying principles of mechanics, and because, in the foundations of Newtonian mechanics, the same importance had not been accorded to gravitational phenomena as to inertial phenomena, which, judged from the standpoint of experience, must be claimed. Gravitation, as a force acting at a distance, is, on the contrary, introduced only as a special force for a limited range of phenomena: and the surprising fact of the equality of inertial and gravitational mass, valid at all times and in all places, receives no further attention. One must, therefore, substitute for the law of inertia a fundamental law which comprises inertial and gravitational phenomena. This can be brought about by a consistent application of the principle of the relativity of all motions, as Einstein has recognized. This is, therefore, the circumstance chosen by Einstein as a nucleus about which to weave his developments.

The theorem of the equality of inertial and gravitational mass, which reflects the intimate connection between inertial and gravitational phenomena, may

be illuminated from another point of view, and thereby discloses *its close relationship* (*vide* page 55) *to the general principle of relativity*.

However much the notion of "absolute space" repelled Newton, he nevertheless believed he had a strong argument, in support of the existence of absolute space, in the phenomenon of centrifugal forces. When a body rotates, centrifugal forces make their appearance. Their presence in a body alone, *without any other visible body being present*, enables one to demonstrate the fact that it is in rotation. Even if the earth were perpetually enveloped in an opaque sheet of cloud, one would be able to establish its daily rotation about its axis by means of Foucault's pendulum-experiment. This peculiarity of rotations led Newton to conclude that absolute motions exist. From the *purely kinematical* point of view, however, the rotation of the earth is not to be distinguished in any way from a translation; in this case, too, we *observe* only the *relative* motions of bodies, and might just as well imagine that all bodies in the universe revolve around the earth. E. Mach has, in fact, affirmed that both events are equivalent, not only kinematically, but also *dynamically*: it must, however, then be assumed that the *centrifugal forces*, which are observed *at the surface of the earth*, would also arise, equal in quantity and similar in their manifestations, from the *gravitational effect* of all bodies in their entirety, if these revolved around the supposedly *fixed* earth (*vide* Note 19).

The justification for this view, which in the first place arises out of the kinematical standpoint, is, in the main, to be sought in the fact, derived from experience, that inertial and gravitational mass are equal. According to the conceptions, which have hitherto prevailed, the centrifugal

forces are called into play by the inertia of the rotating body (or rather by the inertia of the separate points of mass, which continually strive to follow the bent of their inertia, and, therefore, express the tendency to fly off at a tangent to the path in which they are constrained to move). The field of centrifugal forces is, therefore, an *inertial field* (*vide* Note 20). The possibility of *regarding* it equally well as a *gravitational field*—and we do that, as soon as we also *assert* the relativity of rotations dynamically: for we must then assume that the whole of the masses describing paths about the (supposed) fixed body *induce* the so-called centrifugal forces by means of their gravitational action—is founded on the equality of inertial and gravitational mass, a fact which Eötvös has established with extraordinary precision by making use of the centrifugal forces of the rotating earth (*vide* Note 21). *From these considerations one realizes how a general principle of the relativity of all motions simultaneously implies a theory of gravitational fields.*

From these remarks one inevitably gains the impression that a construction of mechanics upon an entirely new basis is an absolute necessity. There is no hope of a satisfactory formulation of the law of inertia without taking into account the *relativity of all motions*, and hence just as little hope of banishing the unwelcome conception of absolute motion out of mechanics; moreover, the discovery of the *inertia of energy* has taught us facts which refuse to fit into the existing system, and necessitate a revision of the foundations of mechanics. The condition which must be imposed at the very outset (cf. page 20) is: Elimination of all actions which are supposed to take place “at a distance” and of all quantities which are not capable of direct observation, out of the fundamental laws; i.e.

the setting-up of a differential equation which comprises the motion of a body under the influence of *both inertia and gravity* and symbolically expresses the relativity of all motions. This condition is completely satisfied by Einstein's theory of gravitation and the general theory of relativity. The sacrifice, which we have to make in accepting them, is to renounce the hypothesis, which is certainly deeply rooted, that all physical events take place in space whose measure-relations (geometry) are given to us *a priori*, independently of all physical knowledge. As we shall see in the following section, the general theory of relativity leads us, rather, to the view that we may regard the metrical conditions in the neighbourhood of bodies as being conditioned by their gravitation. In this way the geometry of the measuring physicist becomes intimately welded with the other branches of physics.

If we compress into a short statement what we have so far deduced out of the fundamental postulates formulated at the beginning, we may say: The postulate of general relativity demands that the fundamental laws be independent of the particular choice of the co-ordinates of reference. But the Euclidean line-element does not preserve its form after any arbitrary change of the co-ordinates of reference. We have, therefore, to substitute in its place the general line-element:

$$ds^2 = \sum_1^4 g_{\mu\nu} dx_\mu dx_\nu.$$

Whereas, then, the postulate of continuity (cf. page 20) seemed to render it only *advisable* not to introduce the narrowing assumptions of the Euclidean determination of measure, the principle of general relativity no longer leaves us any choice.

The reason for so emphasizing the latter principle

—as, indeed, also the postulate that only observable quantities are to occur in physical laws—is not to be sought in any requirement of a merely formal nature, but rather in an endeavour to invest the principle of causality with the authority of a law which holds good in the world of actual physical experience. The postulate of the relativity of all motions receives its true value only in the light of *the theory of knowledge* (Note 22). One must, above all, avoid introducing into physical laws, side by side with observable quantities, hypotheses which are purely fictitious in character, as e.g. the *space* of Newton's mechanics. Otherwise the principle of causality would not give us any real information about causes and effects, i.e. the causal relations of the contents of *direct* experience; which is, presumably, the aim of every physical description of natural phenomena.

§ 5

EINSTEIN'S THEORY OF GRAVITATION

(a) THE FUNDAMENTAL LAW OF MOTION AND THE PRINCIPLE OF EQUIVALENCE OF THE NEW THEORY

AFTER the foregoing remarks we shall be able to proceed to a short account of Einstein's theory of gravitation. Within the limits of the mathematics assumed in this book we shall, of course, only be able to sketch the outlines so far that the assumptions and hypotheses characteristic of the theory come into clear view and that their relation to the two fundamental postulates of the second section becomes manifest. We start out from the fundamental law of motion in classical mechanics, the law of inertia. Since even in the law of inertia all the weaknesses of the old theory come to light, a new fundamental law of motion becomes an absolute necessity for the new mechanics. It is thus natural that we should start building up the new theory from this point. The new law of motion must be a differential law, which, in the first place, describes the motion of a point-mass under the influence of both inertia *and* gravity, and which, secondly, always preserves the same form, irrespective of the system of co-ordinates to which it be referred, so that no system of co-ordinates enjoys a preference to any other. The first condition

arises from the necessity of ascribing the same importance to gravitational phenomena as to inertial phenomena in the new process of founding mechanics—the law must, therefore, also contain terms which denote the gravitational state of the field from point to point; the second condition is derived from the postulate of the relativity of all motion.

A law of this kind exists in the special theory of relativity in the equation of motion of a single point, not subject to any external influence. According to this equation, the path of a point is the “shortest” or “straightest” line (*vide* Note 23)—i.e. the “straight line,” if the line-element ds is Euclidean. Written as an equation of variation this law is:

$$\delta\{ds\} = \delta\{\int \sqrt{-dx^2 - dy^2 - dz^2 + c^2 dt^2}\} = 0.$$

If the principle of the shortest path, which is to be followed in actual motions, be elevated in this form to a general *differential law* for the motion in a *gravitational* field too, with due regard to the principle of the relativity of all motions, the new fundamental law must run as follows:

$$\delta\{ds\} = \delta\{\int \sqrt{g_{11}dx_1^2 + g_{12}dx_1dx_2 + \dots + g_{44}dx_4^2}\} \\ = 0 \dots (I).$$

For only this form of the line-element remains unaltered (invariant) for arbitrary transformations of the x_1, x_2, x_3, x_4 . The factors $g_{11} \dots g_{44}$, which for the present we leave unexplained, occur in it as something essentially new. Now, the extraordinarily fruitful idea that occurred to Einstein was this: Since the new law is to hold for any arbitrary motions whatsoever, thus also for accelerations, such as we perceive in gravitational fields, we must make the gravitation field, in which the

observed motion takes place, responsible for the occurrence of these ten factors $g_{\mu\nu}$. These ten co-efficients $g_{\mu\nu}$ which will, in general, be functions of the variables $x_1, \dots, x_4$, must, if the new fundamental law is to be of use, be able to be brought into such relationship to the gravitational field, in which the motion takes place, that they are determined by the field, and that the motion *described* by equation (1) coincides with the *observed* motion. This is actually possible. (The $g_{\mu\nu}$'s are the gravitational potentials of the new theory, i.e. they take over the part played by the one gravitational potential in Newton's theory, without, however, having the special properties, which according to our knowledge a potential has, in addition.)

Corresponding to the measure-relations of a space-time manifold based upon the line-element :

$$ds^2 = \sum_1^4 g_{\mu\nu} dx_\mu dx_\nu,$$

which is now placed at the foundation of mechanics by virtue of the relativity of all motions, the remaining physical laws must also be so formulated that they remain independent of the accidental choice of the variables. Before we enter into this more closely, the distinguishing features of the theory of gravitation characterized by equation (1) will be considered in greater detail.

The postulate of the new theory, that the laws of mechanics are only to contain statements about the relative motions of bodies, and that, in particular, the motion of a body under the action of the attraction of the remaining bodies is to be symbolically described by the formula :

$$\delta \left\{ \sqrt{\sum_1^4 g_{\mu\nu} dx_\mu dx_\nu} \right\} = 0,$$

is fulfilled in Einstein's theory by a physical hypothesis concerning the nature of gravitational phenomena, which he calls the *hypothesis* or *principle* (respectively) of *equivalence* (*vide* Note 24). This asserts the following :

Any change, which an observer perceives in the passing of any event to be due to a gravitational field, would be perceived by him in exactly the same way, if the gravitational field were not present, provided that he—the observer—makes his system of reference move with the acceleration which was characteristic of the gravitation at his point of observation.

For, if the variables x, y, z, t in the equation of motion

$$\delta\{ds\} = \delta\{\sqrt{-dx^2 - dy^2 - dz^2 + c^2dt^2}\} = 0$$

of a point-mass moving uniformly and rectilinearly (i.e. uninfluenced by gravity) be subjected to any transformation corresponding to the change of the x, y, z, t into the co-ordinates x_1, x_2, x_3, x_4 of a system of reference which has any accelerated motion whatsoever with regard to the initial system x, y, z, t ; then, in general, coefficients $g_{\mu\nu}$ will occur in the transformed expression for ds , and will be functions of the new variables $x_1, \dots, x_4$, so that the transformed equation will be :

$$\delta\{\sqrt{g_{11}dx_1^2 + g_{12}dx_1dx_2 + \dots + g_{44}dx_4^2}\} = 0.$$

Taking into account the extended region of validity of this equation, one will be able to regard the $g_{\mu\nu}$ which arise from the accelerational transformation (*vide* Note 25) just as well, as due to the action of a gravitational field, which asserts its existence in effecting just these accelerations. *Gravitational problems thus resolve into the general science of motion of a relativity-theory of all motions.*

By thus accentuating the equivalence of gravita-

tional and accelerational events, we raise the fundamental fact, that *all* bodies in the gravitational field of the earth fall with *equal* acceleration, to a fundamental assumption for a new theory of gravitational phenomena. This fact, in spite of its being reckoned amongst the most certain of those gathered from experience, has hitherto not been allotted any position whatsoever in the foundations of mechanics. On the contrary, the Galilean law of inertia makes an event which had never been actually observed (the uniform rectilinear motion of a body, which is not subject to external forces) function as the main-pillar amongst the fundamental laws of mechanics. This brought about the strange view that inertial and gravitational phenomena, which are probably not less intimately connected with one another than electric and magnetic phenomena, have nothing to do with one another. The phenomenon of inertia is placed at the base of classical mechanics as the fundamental property of matter, whereas gravitation is only, as it were, introduced by Newton's law as one of the many possible forces of nature. The remarkable fact of the equality of the inertial and gravitational mass of bodies only appears as an accidental coincidence.

Einstein's principle of equivalence assigns to this fact the rank to which it is entitled in the theory of motional phenomena. The new equation of motion (I) is intended to describe the relative motions of bodies with respect to one another under the influence of *both* inertia *and* gravity. The gravitational and inertial phenomena are amalgamated in the *one* principle that the motion take place in the geodetic line ($\delta \int ds = 0$). Since the element of arc

$$ds = \sqrt{\sum_1^4 g_{\mu\nu} dx_\mu dx_\nu}$$

preserves its form after any arbitrary transformation of the variables, all systems of reference are *equally* justified as such, i.e. there is none which is more important than any other.

The most important part of the problem, with which Einstein saw himself confronted, was the setting-up of differential equations for the gravitational potentials $g_{\mu\nu}$ of the new theory. With the help of these differential equations, the $g_{\mu\nu}$'s were to be unambiguously calculated (i.e. as single-valued functions) from the distribution of the quantities exciting the gravitational field; and the motion (e.g. of the planets) which was described, according to equation (1) by inserting these values for the $g_{\mu\nu}$'s, had to agree with the observed motion, if the theory was to hold true. In setting up these differential equations for the gravitational potentials $g_{\mu\nu}$ Einstein made use of hints gathered from Newton's theory, in which the factor which excites the field in Poisson's equation $\Delta\phi = -4\pi\rho$ for the Newtonian gravitational potential (viz. the factor represented by ρ , the density of mass in this equation) is put proportional to a *differential expression of the second order*. This circumstance prescribes, as it were, the method of building up these equations, taking for granted that they are to assume a form similar to that of Poisson's equation.

In conformity with the deepened meaning we have assigned to the mutual relation between inertia and gravity, as well as to the connection between the inertia and latent energy of a body, we find that ten components of the quantity which determines the "*energetic*" state at any point of the field, and which was already introduced by the special theory of relativity as "*stress-energy-tensor*," duly make their appearance in place of the density of mass ρ , in Poisson's equation.

Concerning the differential expressions of the second order in the $g_{\mu\nu}$'s which are to correspond to the $\Delta\phi$ of Poisson's equation, Riemann has shown the following: the measure-relations of a manifold based on the line-element

$$ds^2 = \sum_1^4 g_{\mu\nu} dx_\mu dx_\nu,$$

are in the first place determined by a differential expression of the fourth degree (the Riemann-Christoffel Tensor), which is independent of the arbitrary choice of the variables $x_1, \dots, x_4$ and from which all other differential expressions which are likewise independent of the arbitrary choice of the variables $x_1, \dots, x_4$ and only contain the $g_{\mu\nu}$'s and their derivatives, can be developed (by means of algebraical and differential operations). This differential expression leads unambiguously, i.e. in only one possible way, to ten differential expressions in the $g_{\mu\nu}$'s. And now, in order to arrive at the required differential equations, Einstein puts these ten differential expressions proportional to the ten components of the stress-energy-tensor, regarding the latter ten as the quantities exciting the field. He inserts the gravitational constant as the constant of gravitation. **These differential equations for the $g_{\mu\nu}$'s, together with the principle of motion given above, represent the fundamental laws of the new theory.** To the first order they, in point of fact, lead to those forms of motion, with which Newton's theory has familiarized us (*vide* Note 26). More than this, without requiring the addition of any further hypothesis, they mathematically account for the only phenomenon in the theory of planetary motion which could not be explained on the Newtonian theory, viz. the occurrence of the remainder-term in the expression

for the motion of Mercury's perihelion. Yet we must bear in mind that there is a certain arbitrariness in these hypotheses just as in that made for the fundamental law of motion. Only the careful elaboration of the new theory in all its consequences, and the experimental testing of it will decide whether the new laws have received their final forms.

Since the formulæ of the new theory are based upon a space-time-manifold, the line-element of which has the general form

$$ds = \sqrt{\sum_1^4 g_{\mu\nu} dx_\mu dx_\nu},$$

all other physical laws, in order to bring the general theory of relativity to its logical conclusion, must receive (see p. 46) a form which, in agreement with the new measure-conditions, must be independent of the arbitrary choice of the four variables x_1, x_2, x_3, x_4 .

Mathematics has already performed the preliminary work for the solution of this problem in the calculus of absolute differentials; Einstein has elaborated them for his particular purposes (in his essay "Concerning the formal foundations of the general theory of relativity*"); Gauss invented the calculus of absolute differentials in order to study those properties of a surface (in the theory of surfaces) which are not affected by the position of the surface in space nor by inelastic continuous deformations of the surface (deformations without tearing), so that the value of the line-element does not alter at any point of the surface. As such properties depend upon the *inner* measure-relations of the surface only, one avoids

* "Über die formalen Grundlagen der allgemeinen Relativitätstheorie," *Sitz. Ber. d. Kgl. Preuss. Akad. d. Wiss.*, xli., 1916, S. 1080.

referring, in the theory of surfaces, to the usual system of co-ordinates, i.e. one avoids reference to points which do not themselves lie on the surface. Instead of this, every point in the surface is fixed, by covering the surface with a net-work, consisting of two intersecting arbitrary systems of curves, in which each curve is characterized by a parameter ; every point of the surface is then unambiguously, i.e. singly, defined by the two parameters of the two curves (one from each system) which pass through it. According to this view of surfaces, a cylindrical envelope and a plane, for instance, are not to be regarded as different configurations : for each can be unfolded upon the other without stretching, and accordingly the same planimetry holds for both—a criterion that the inner measure-relations of these two manifolds are the same (*vide* Note 27). The general theory of relativity is based upon the same view ; but now *not* as applied to the two-dimensional manifold of surfaces, but with respect to the *four-dimensional* space-time manifold. As the four space-time variables are devoid of all physical meaning, and are only to be regarded as four parameters, it will be natural to choose a representation of the physical laws, which provides us with differential laws which are independent of the chance choice of the x_1, x_2, x_3, x_4 ; this is what is done by the calculus of absolute differentials. The results of the preceding paragraphs, the far-reaching consequences of which can be fully recognized only by a detailed study of the mathematical developments involved, may be summarized as follows :

A mechanics of the relative motions of bodies, which is in harmony with the two fundamental postulates of continuity and relativity, can be built up only on a fundamental law of motion

that preserves its form independently of the kind of motion the system is undergoing. An available law of this kind is given if we raise the law of motion along a geodetic line, which, in the special theory of relativity, holds only for a body moving under no forces, to the rank of a general differential law of the motion in the gravitational field, too. In this general law, we must, it is true, give the line-element of the orbit of the moving body the general form :

$$ds = \sqrt{\sum_1^4 g_{\mu\nu} dx_\mu dx_\nu},$$

at which we arrive in the second section, using as our basis the two fundamental postulates. The new functions $g_{\mu\nu}$ that now occur may be interpreted as the potentials of the gravitational field, if we take our stand on the hypothesis of equivalence. To calculate the quantities $g_{\mu\nu}$ from the factors determining the gravitational field, namely, matter and energy, it immediately suggests itself to us to assume a system of differential equations of the second order, that are built up analogously to Poisson's differential equation for the Newtonian gravitational potential. These differential equations, together with the fundamental law of motion, represent the fundamental equations of the new mechanics and the theory of gravitation.

Since the new theory uses the generalized curvilinear co-ordinates x_1, x_2, x_3, x_4 , and not the Cartesian co-ordinates of Euclidean geometry, all the other physical laws must also receive a general form that is independent of the special choice of co-ordinates. The mathematical instrument for remoulding these formulæ is given by the general calculus of differentials.

This theory, which is built up from the most

general assumptions, leads, for a first approximation, to Newton's laws of motion. Wherever deviations from the theory hitherto accepted reveal themselves, we have possibilities of testing the new theory experimentally. Before we turn to this question, let us look back, and become clear as to the attitude which the general theory of relativity compels us to adopt towards the various questions of principle we have touched upon in the course of this essay.

(b) RETROSPECT

I. The conceptions "inertial" and "gravitational" (heavy) mass no longer have the absolute meaning which was assigned to them in Newton's mechanics. The "mass" of a body depends, on the contrary, exclusively upon the presence and relative position of the remaining bodies in the universe. The equality of inertial and gravitational mass is put at the head of the theory as a rigorously valid principle. The hypothesis of equivalence herein supplements the deduction of the special theory of relativity, that all energy possesses inertia, by investing all energy with a corresponding gravitation. It becomes possible—on the basis (be it said) of certain special assumptions into which we cannot enter here—to regard rotations unrestrictedly as relative motions too, so that the centrifugal field around a rotating body can be interpreted as a gravitational field, produced by the revolution of all the masses in the universe about the non-rotating body in question. In this manner mechanics becomes a perfectly general theory of relative motions. As our statements are concerned only with observations of relative motions, the new mechanics fulfils the postulate

that in physical laws observable things only are to be brought into causal connection with one another. It also fulfils the postulate of continuity ; since the new fundamental laws of mechanics are differential laws, which contain only the line-element ds and no finite distances between bodies.

2. The principle of the constancy of the velocity of light *in vacuo*, which was of particular importance in the special theory of relativity, is no longer valid in the general theory of relativity. It preserves its validity only in regions in which the gravitational potentials are constant, finite portions of which we can never meet with in reality. The gravitational field upon the earth's surface is certainly so far constant that the velocity of light, within the limits of accuracy of our measurements, had to appear to be a universal constant in the results of Michelson's experiments. In a gravitational field, however, in which the gravitational potentials vary from place to place, the velocity of light is not constant ; the geodetic lines, along which light propagates itself, will thus in general be curved. The proof of the curvature of a ray of light, which passes by in close proximity to the sun, offers us one of the most important possibilities of confirming the new theory.

3. The greatest change has been brought about by the general theory of relativity in our conceptions of space and time.*

According to Riemann the expression for the line-element, viz.

$$ds^2 = \sum_1^4 g_{\mu\nu} dx_\mu dx_\nu,$$

* This aspect of the problem has been treated with particular clearness and detail in the book "Raum und Zeit in der gegenwärtigen Physik," by Moritz Schlick, published by Jul. Springer, Berlin. The Clarendon Press has published an English rendering under the title: "Space and Time in Contemporary Physics."

determines, in our case, the measure-relations of the continuous space-time manifold ; and according to Einstein the coefficients $g_{\mu\nu}$ of the line-element ds have, in the general theory of relativity, the significance of *gravitational potentials*. Quantities, which hitherto had only a purely geometrical import, for the first time became animated with physical meaning. It seems quite natural that gravitation should herein play the fundamental part, viz. that of predominating over the measure-laws of space and time. For there is no physical event in which it does not co-operate, inasmuch as it rules wherever matter and energy come into play. Moreover, it is the only force, according to our present knowledge, which expresses itself quite independently of the physical and chemical constitution of bodies. It therefore without doubt occupies a unique position, in its outstanding importance for the construction of a physical picture of the world.

According to Einstein's theory, then, gravitation is the "inner ground of the metric relations of space and time" in Riemann's sense (*vide* the final paragraph of Riemann's essay "On the hypotheses which lie at the bases of geometry" quoted on p. 29). If we uphold the view that the space-time manifold is continuously connected, its measure-relations are not then already contained in its definition as being a continuous manifold of the dimensions "four." These have, on the contrary, yet to be gathered from experience. And it is, according to Riemann, the task of the physicist finally to seek the inner ground of these measure-relations in "binding forces which act upon it." Einstein has discovered in his theory of gravitation a solution to this problem, which was presumably first put forward in such clear terms by Riemann.

At the same time he gives an answer to the question of the true geometry of physical space, a question which has exercised physicists for the last century, —but an answer, it is true, of a sort quite different from that which had been expected.

The alternative, Euclidean or non-Euclidean geometry, is not decided in favour of either one or the other; but rather space, as a physical thing with given geometrical properties, is banished out of physical laws altogether: just as ether was eliminated out of the laws of electrodynamics by the Lorentz-Einstein special theory of relativity. This, too, is a further step in the sense of the postulate that only observable things are to have a place in physical laws. The inner ground of metric relations of the space-time manifold, in which all physical events take place, lies, according to Einstein's view, in the gravitational conditions. Owing to the continual motion of bodies relatively to one another, these gravitational conditions are continually altering; and, therefore, one cannot speak of an invariable given geometry of measure or distance—whether Euclidean or non-Euclidean. As the laws of physics preserve their form in the general theory of relativity, independent of how the four variables $x_1, \dots, x_4$ may chance to be chosen, the latter have no absolute physical meaning. Accordingly x_1, x_2, x_3 , for instance, will not in general denote three distances in space which can be measured with a metre rule, nor will x_4 denote a moment of time which can be ascertained by means of a clock. The four variables have only the character of numbers, parameters, and do not immediately allow of an objective interpretation. Time and space have, therefore, not the meaning of real physical things in the description of the events of physical nature.

And yet it seems as if the new theory may even be able to give a definite answer in favour of one or other of the above alternatives, if, namely, we postulate their validity for the world as a whole. The application of the formulæ of the new theory to the world as a whole at first led to the same difficulties as those revealed in classical mechanics. Boundary conditions for what is infinitely distant could not be set up entirely satisfactorily and at the same time satisfy the condition of general relativity. Yet Einstein * succeeded in extending the differential equations for the gravitational potentials $g_{\mu\nu}$ in such a way that it became possible to apply his theory of gravitation to the universe. The difficulties that arose for the boundary conditions at infinity here vanished, for an extraordinarily interesting reason. For it was shown that in these new formulæ a space that is filled uniformly with matter *which is at rest* would, to a first approximation, be built up like an, indeed, unbounded, but *finitely* closed space, so that boundary conditions would not appear at all for infinity. Even if the assumptions that would lead to this result are not fulfilled in the world, yet it must be remembered that the velocities of matter as ascertained in the case of the stars are extraordinarily small compared with the velocity of light which we now take as our unit. Nor does the distribution of the matter so far show, in the main, irregularities sufficient to place Einstein's view of a stationary, uniformly-filled world quite out of the realm of possible truth. Thus this deduction of the theory would answer our above alternative in this sense: the geometry that we must use as our basis of spatial happening is, indeed, neither Euclidean nor

* "Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie" *Sitz. Ber. d. Preuss. Akad. der Wiss.*, 1917, p. 142.

non-Euclidean, but, as stated above, conditioned by the gravitational states from place to place. But a world built up according to the simplest scheme would in the new theory behave on the whole like a finite closed manifold, that is, as if it were non-Euclidean. Even if this result is only of theoretical importance for the present, since the stellar system that we see around us does not fulfil Einstein's assumptions—in particular, the scarcely-to-be-doubted flattening of the Milky Way is not compatible with these simple assumptions—and since we have at present no knowledge of the stellar systems outside the Milky Way, yet this aspect of the theory opens up undreamed-of perspectives for our view of the world as a whole.

4. The gravitational theory, which emerges out of the general theory of relativity, is, in contradistinction to the Newtonian theory, built up, not upon an elementary law of the *gravitational forces*, but upon an elementary law of the *motion* of a body in the *gravitational field*. Consequently, the expressions which would be *interpreted* as gravitational forces in the new theory play only a minor part in the building-up of the theory (as indeed the conception of force in mechanics altogether is to be regarded as only an auxiliary or derived conception, if we regard it as the object of mechanics to give a flawless *description* of the motions occurring in physical events).

Nor does Einstein's theory endeavour to explain the *nature* of gravitation; it does not seek to give a mechanical model, which would symbolize the gravitational effect of two masses upon one another. This is what the various theories involving ether-impulses attempted to do, by freely using hypothetical quantities which had never been actually observed, such as ether-atoms. It is very doubtful whether such endeavours will ever lead to a satis-

factory theory of gravitation. For, the difficulties of Newton's mechanics are not contained only in the fact that it formulates the law of gravitation as a law of forces acting at a distance. Two much more serious points are: first, that the close relationship existing between inertial and gravitational phenomena receives no recognition whatsoever, although Newton was already aware of the fact that inertial and gravitational mass are equal; and second, that Newton's mechanics does not present us with a theory of the relative motions of bodies, although we only observe relative motions of bodies with respect to one another. Re-moulding Newton's law of gravitational force, in order to make the attraction of matter more feasible, would therefore not have helped us finally to a satisfactory theory of the phenomena of motion (*vide* Note 28).

What distinguishes the Newtonian theory, above all, is the extraordinary simplicity of its mathematical form. Classical mechanics, which is built up on Newton's initial construction, will, for this reason, never lose its importance as an excellent mathematical theory for arithmetically following the observed phenomena of motion.

Einstein's theory, on the other hand, as far as the uniformity of its conceptual foundations is concerned, satisfies all the conditions for a physical theory. The fact that (by abandoning the Euclidean measure of distance) it cuts its connection with the familiar representation by means of Cartesian coordinates, will not be felt to be a disturbing factor, as soon as the analytical appliances, which have been called into use as a help, have been more generally adopted. This mathematical elaboration of the theory at the same time gives to the astronomer the task of testing the theory experimentally in those phenomena in which measurable deviations from the results of the classical theory arise.

§ 6

THE VERIFICATION OF THE NEW THEORY BY
ACTUAL EXPERIENCE

AS far as can be seen at present, there are three possible experiments for verifying Einstein's theory of gravitation; all three can be performed only by the agency of astronomy. One of them—arising from the deviation of the motion of a material point in the gravitational field according to Einstein's theory, as compared with that required by Newton's theory—has already decided in favour of the new theory: not less so one of the other two that arise through a combination of electromagnetic and gravitational phenomena.

Since the sun far exceeds all other bodies of the solar system in mass, the motion of each particular planet is primarily conditioned by the gravitational field of the sun. Under its action the planet describes, according to Newton's theory, an ellipse (Kepler's law), the major axis of which—defined by connecting the point of its path nearest the sun (perihelion) with the farthest point (aphelion)—is at rest, relative to the stellar system. Upon this elliptic motion of a planet there are superimposed more or less considerable influences (disturbances) due to the remaining planets, which do not, however, appreciably alter the elliptic form; these influences partly only call forth periodical fluctua-

tions in the defining elements of the initial ellipse (i.e. major axis, eccentricity, etc.), partly cause a continual increase or decrease of the latter. In this second kind of disturbance are also to be classed the slow *rotation* of the major axis, and consequently also of the corresponding perihelion, relative to the stellar system; which has been observed in the case of all planets. For all the larger planets, the *observed* motions of the perihelion agree with those *calculated* from the disturbing effects (except for small deviations which have not been definitely established, as in the case of Mars); on the other hand, in the case of Mercury the calculations give a value which is too small by $43''$ per 100 years. Hypotheses of the most diverse description have been evolved to explain this difference; but all of them are unsatisfactory. They oblige one to resort to still unknown masses in the solar system: and, as all the searches for masses large enough to explain this anomalous behaviour of Mercury prove fruitless, one is compelled to make assumptions about the distribution of these hypothetical masses, in order to excuse their *invisibility*. In view of these circumstances, there is no shade of probability in these hypotheses.

According to Einstein's theory, a planet, at the distance of Mercury for instance, moves, under the action of the sun's attraction, along the "straightest path," according to the equation:

$$\delta\{ds\} = \delta\{\sqrt{g_{11}dx_1^2 + g_{12}dx_1dx_2 + \dots + g_{44}dx_4^2}\} = 0.$$

The $g_{\mu\nu}$'s can be derived from the differential equations, which were given for them above, and which result from the *assumed sole presence of the sun and the planet being regarded as a mass concentrated at a point*. Einstein's developments give the

ellipse of Kepler too as a first approximation for the path of the planet: at a higher degree of approximation, however, it is found that the radius vector from the sun to the planet, between two consecutive passages through perihelion and aphelion, sweeps out an angle, which is about $0.05''$ greater than 180° ; so that, for each complete revolution of the planet in its path, the major axis of the path—i.e. the straight line connecting perihelion with aphelion—turns through about $0.1''$ in the sense in which the path is described. Therefore, in 100 years—Mercury completes a revolution in 88 days—the major axis will have turned through $43''$. The new theory, therefore, actually explains the hitherto inexplicable amount, 43 seconds per 100 years, in the motion of Mercury's perihelion, merely from the effect of the sun's gravitation. (The deviations due to such disturbances would only differ very inappreciably from the values obtained by Newton's theory in the case of the remaining planets.) The only arbitrary constant which enters into these calculations is the gravitational constant which figures in the differential equations for the gravitational potentials $g_{\mu\nu}$, as has already been mentioned on page 50. This achievement of the new theory can scarcely be estimated too highly.

The reason that a measurable deviation from the results according to Newton's theory occurs in the case of Mercury, the planet nearest to the sun, but not in the case of the planets more distant from the sun, is that this deviation decreases rapidly with increasing distance of the planet from the sun, so that it already becomes imperceptible at the distance of the earth. In the case of Venus, the eccentricity of the path is, unfortunately, so small, that it scarcely differs from a circle: and the

position of the perihelion can, therefore, only be determined with great uncertainty.

Of the other two possibilities of verifying the theory, one arises from the influence of gravitation upon the time an event takes to pass. How such an influence can come about, will be evident from the following example: According to the new theory, an observer cannot immediately distinguish whether a change, which he observes during the passage of a certain event, is due to a gravitational field or to a corresponding acceleration of his place of observation (his system of reference). Let us assume an invariable gravitational field, denoted by parallel lines of force in the negative direction of the z -axis, and having a constant value γ for the acceleration with which all bodies in the field fall (i.e. characterized by conditions which approximately exist on the surface of the earth). According to Einstein's theory, any event will take place in this field in just the same way as it appears to occur when referred to a co-ordinate system which has an acceleration γ in the positive direction of the z -axis. Now if a ray of light, the time of oscillation of which is ν_1 , travels from a point A—which is to be conveniently supposed at rest relatively to the corresponding co-ordinate system at the moment of departure of the ray—in the direction of the z -axis for a distance h to a point B, then an observer at B will, owing to his own acceleration, γ , have attained a velocity $\gamma \cdot h/c$ at the instant the ray of light reaches him (c denotes the velocity of light). According to the usual Doppler Principle, he will assign a time of oscillation $\nu_2 = \nu_1 (1 + \gamma \cdot h/c^2)$ to the ray of light as a first approximation, instead of ν_1 . If we transfer the same event to the equivalent gravitational field, this result assumes the following form: The time of oscillation ν_2 of a ray of light

at a place B, the gravitational potential of which differs from that of a place A by the amount $\pm \phi$, is connected with the time of oscillation there observed by the relation :

$$\nu_2 = \nu_1 (1 + \phi/c^2),$$

according to the principle of equivalence of Einstein's theory of gravitation.

This special case shows how the duration of an event is to be understood as being dependent upon the gravitational condition.

Moreover, one can regard every vibrating system (which emits a spectral line) as a *clock*, the motion of which, according to the investigation made just above, depends upon the gravitational potentials of the place where it is stationed. This same "clock" will have a different time of oscillation at another place in the field according to the gravitational potential, i.e. it will go at a different rate. Consequently, a particular line in the spectrum of the light which comes from the *sun*, e.g. an Fe-line (iron), must appear to be shifted in comparison with the corresponding line as produced by a source of light (arc-lamp) on the *earth*; the gravitational potential at the surface of the sun has, corresponding to the latter's great mass, a different value from that at the surface of the earth, and a definite time of oscillation (colour) is characterized in the spectrum by a definite position (Fraunhofer line). It has not yet been possible to observe this effect, which amounts to about $0.008\text{\AA}^*$ for a wave-length of $400\mu\mu$ with certainty. For the conditions of emission of the light from the sun's surface have not yet been sufficiently investigated, and the systematic errors in the wave-lengths in

* $\text{\AA} = \text{\AA}ngstr\ddot{o}m \text{ unit} = 10^{-8} \text{ cm.}$

the light from the source used for comparison on the earth, the arc-lamp, are not yet sufficiently known to allow the negative results of observation hitherto obtained to be regarded as giving binding decisions. This is the more true inasmuch as in the case of the fixed stars there are, doubtless, signs of the presence of a gravitational shift of the spectral lines (*vide* the closing essay of this book). It is a particularly important task of astronomy to establish this effect with certainty, for this gravitational displacement of the spectral lines is a direct consequence of the hypothesis of equivalence, and does not assume the other hypotheses of the theory such as, for example, the differential equations of the gravitational field.

The third and *particularly important inference from Einstein's theory* is the dependence of the velocity of light upon the gravitational potential, and the resultant curvature (based upon Huygens' principle) of a ray of light in passing through a gravitational field. The theory asserts that a ray of light, coming e.g. from a fixed star, and which passes in close proximity to the sun, has a *curved path*. As a consequence of this *curvature*, the star must appear displaced from its true position in the heavens by an amount which attains the value $1.7''$ at the edge of the sun's disc, and decreases in proportion to the distance from the centre of the sun. But since a ray of light which comes from a fixed star and passes by the sun can be caught only when the light of the sun, which overpowers all else by its brilliancy, is intercepted before its entrance into our atmosphere, only the rare moments of a total eclipse come into account for this observation and for the solution of the problem. The solar eclipse of 29th May, 1919, during which photographs were taken at two widely-

separated stations, for the purpose of this test, has, as far as the results of measurement allow us to pass definite judgment, decided in favour of the general theory of relativity.*

The experimental verification of Einstein's theory of gravitation has thus not reached completion. But if, in spite of this, the theory can, even at this early stage, justly claim general attention, the reason is to be found in the unusual unity and logical structure of the ideas underlying it. In truth, it solves, at one stroke, all the riddles, concerning the motions of bodies, which have presented themselves since the time of Newton, as the result of the conventional view about the meaning of space and time in the physical description of natural phenomena.

* The results were made public at the meeting of the Royal Society on the 6th Nov., 1919.—H. L. B.

APPENDIX

Note 1 (p. 4). So long as the *universal* significance of the velocity of light remained unknown, two conjectures were possible in the question as to whether, under certain circumstances, the motion of the source of light would make itself observable in the velocity of propagation of light. It might be surmised that the velocity of the source simply added itself to that velocity of light which is *characteristic for the propagation of the light* from a source at rest. Or, it might be conceived that the motion of the source has no influence at all on the velocity of the light emitted by it. In the second case it was imagined that the source of light only excites the periodically changing states of the luminiferous ether, which is at rest, that is, which does not share in the motion of the matter (source of light), and that these states then propagate themselves with a *velocity that is characteristic of the ether*, and with a velocity that makes these states perceptible to us as light waves. This view had finally apparently won the day. It was the advent of the special theory of relativity and the quantum hypothesis that made this view impossible. For the special theory of relativity, in robbing the assertion: "the ether is at rest" of its significance, since we may arbitrarily define any system as being at rest in the ether, as far as uniform translations are concerned, and in depriving the luminiferous ether of its existence, deprived light-waves of their carrying or

transmitting medium. The quantum hypothesis, in raising light-quanta to the rank of self-supporting individuals, deprived the *velocity of light* of its character as a constant that is characteristic of the *ether*. Thus, our view of light-quanta again leads to a kind of emission theory of light. According to classical mechanics it would have been typical of a theory of emission for the velocity of the source in motion to have added itself to the velocity of the light from the source at rest. We thus revert to the conjecture which we quoted first above. Now, such a superposition of velocities would necessarily cause quite remarkable phenomena in the case of spectroscopic binary stars (de Sitter, "Phys. Zeitschrift," 14, 429). For if two stars move in circular Kepler orbits around each other, and if our line of sight lies in the common plane of the orbits, then we should necessarily perceive the following: if $2T$ is the time of revolution of the system, ν the orbital velocity of the one (bright) component, Δ the distance of the whole system from the earth, and, finally, c , the velocity in vacuo of the light from the source which is at rest, then the velocity of light at the epoch of greatest positive velocity in the direction of vision is $c + \nu$, and $c - \nu$ in the other direction, respectively. Consequently the time-interval between two such successive positions would have the values $T + \frac{2\nu\Delta}{c^2}$

and $T - \frac{2\nu\Delta}{c^2}$ alternately, for the observer on the earth, as a simple calculation shows. Since, on account of the gigantic distances between the fixed stars, the member $\frac{2\nu\Delta}{c^2}$ may become very great, indeed, greater than T , we should be able to observe definite anomalies in the case of the spectro-

scopic binaries. For the time intervals between two such successive epochs in the orbit should be able to contract to nil, indeed, even become negative, and we should not be able to interpret the measured Doppler effects by means of motions in the Kepler ellipses. In reality, however, these anomalies have never manifested themselves. Observation of these very sensitive subjects of test (spectroscopic binaries) teaches us that the motion of the source of light does not make itself remarked in the propagation of the light. This renders our first view likewise untenable. The special principle of relativity, alone in postulating the constancy of the velocity of light, and in putting forward a new addition theorem of velocities, has led us to an attitude in this question that is free from inner contradictions and compatible with experience. (Cf. Note 2.)

Note 2 (p. 5). There are essentially two fundamental optical experiments on which our view of the distinctive significance of the velocity of light in physical nature is founded: Fizeau's experiment concerning the velocity of light in flowing water, and the Michelson-Morley experiment. Aberration, on the other hand, has nothing to do directly with the question whether it is possible to prove by means of optical experiments in the laboratory a motion of the earth relative to the ether. The aberration in the case of stars states merely that the motion of the earth *relatively* to the star under consideration *changes* periodically in the course of a year. If, however, we hold the view that an all-pervading ether is the carrier for the propagation of the light, the phenomenon of aberration may be satisfactorily explained only if we assume that this ether does not participate in the motion of the earth.

Fizeau's experiment was designed to decide

finally whether moving matter influences the ether and to determine the value of the velocity of light in moving matter with respect to the observer. Michelson and Morley repeated the experiment in the following improved form. A beam of light from a source on the earth is sent through a U-shaped tube, through which water flows, in the direction of both limbs. After each part of the beam has traversed the flowing water, the one in the direction of the current, the other contrary to it, the two beams, are made to interfere. The light and the water move in the same direction in the one limb, and oppo-

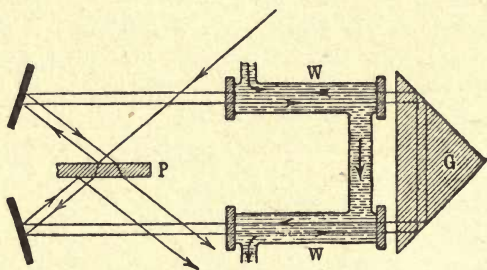


FIG. 1.

sitely in the other. Now, there immediately appear to be two possibilities. Either the water that flows with a velocity v with respect to the walls of the tube drags along the carrier that effects the transmission of the light, namely the ether; in this case, the velocity of the light is $\frac{c}{n} + v$ in the one limb, and $\frac{c}{n} - v$ in the other, for, on account of the coefficient of refraction n of the water, $\frac{c}{n}$ is the velocity of light in resting water. Or, the motion of the water has no influence at all on the

ether which transmits light and which permeates the water. In this case the velocity of light is $\frac{c}{n}$

in both limbs. According, as the one or the other of these two assumptions is valid, the interference fringes would have to become displaced or remain at rest when the direction of the current is reversed. The experiment decided in favour of *neither* of these possibilities. The interference fringes did, indeed, become shifted, not to the expected amount, however, but only to an amount that would result if the ether assumes the velocity $v\left(1 - \frac{1}{n^2}\right)$ in water,

and not the full value v . This value of the convection of the ether is called *Fresnel's convection coefficient*. Yet this term is capable of being misunderstood inasmuch as in the electrodynamics developed by Lorentz, the result of Fizeau's experiment speaks in favour of an ether that is *absolutely at rest*, and the so-called convection coefficient is only a consequence of the structure of matter, in particular of the interaction between electrons and matter, a question into which we cannot enter here. At any rate, at the time preceding the Michelson-Morley experiment aberration, as well as Fizeau's experiment, appeared to speak in favour of an ether that was absolutely at rest.

Now, the Michelson-Morley experiment was to establish the existence of the current of ether (ether "wind") through which the earth continually moves, since the ether is supposed not to participate in the motion of the earth. The scheme of the experiment is as on p. 74.

A ray of light, starting out from L, traverses the course $LP + PS_1 + S_1P + PF_1$: here S_1 and S_2 are two mirrors, on to which the ray falls perpendicularly; P is a glass plate that reflects one half of

the light and allows the remainder to pass through ; F is the telescope of the observer. Another ray of light traverses the course $LP + PS_2 + S_2P + PF$. Let $PS_1 = PS_2 = l$. Further, let FS_1 be in the direction of the earth's motion. Our assumption is that the ether does not share in the earth's motion. Let the velocity of the earth be q . Then the

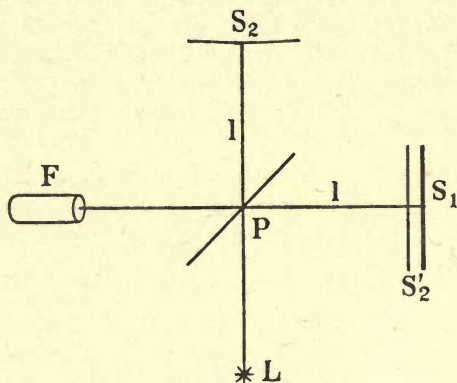


FIG. 2.

velocity of the light relative to the instrument (earth) is as follows in the directions specified :

$$\begin{array}{llll}
 PS_1 : c - q, & \text{thus the light-time for the journey is} & \frac{l}{c - q} \\
 S_1P : c + q, & & \frac{l}{c + q} \\
 PS_2 : \sqrt{c^2 - q^2}, & & \\
 S_2P : \sqrt{c^2 - q^2}, & & \left. \begin{array}{l} \\ \\ \end{array} \right\} \frac{l}{\sqrt{c^2 - q^2}}
 \end{array}$$

Consequently, the course $PS_1 + S_1P$ is traversed in the time

$$t_1 = l \left(\frac{1}{c - q} + \frac{1}{c + q} \right) = \frac{2l}{c} \left(1 + \frac{q^2}{c^2} \right)$$

and the course $PS_2 + S_2P$ in the time

$$t_2 = \frac{2l}{\sqrt{c^2 - q^2}} = \frac{2l}{c} \left(1 + \frac{1}{2} \frac{q^2}{c^2} \right).$$

The difference of these two times is

$$(t_1 - t_2)_1 = \frac{l}{c} \cdot \frac{q^2}{c^2}$$

If we exchange the positions of S_1 and S_2 by turning the whole apparatus through 90° , then

$$(t_1 - t_2)_2 = - \frac{l}{c} \cdot \frac{q^2}{c^2}.$$

If we make these two rays of light interfere at F then, when the apparatus is turned through 90° , the interference fringes should become shifted. The amount of this displacement may easily be calculated. If we denote by τ the vibration frequency of the light-ray used in the experiment, then $c\tau = \lambda$ is the corresponding wave-length. Thus, expressed in fractions of the interval between the fringes, the expected displacement becomes equal to

$$\frac{(t_1 - t_2)_1 - (t_1 - t_2)_2}{\tau} = \frac{2l}{\lambda} \cdot \frac{q^2}{c^2}.$$

By causing the light to be reflected many times $2l$ was magnified to such an extent that $\frac{2l}{\lambda}$ became of the order 10^7 . If, for example, $2l = 30$ metres $= 30 \cdot 10^2$ cms., $\lambda = 6 \cdot 10^{-5}$ cms. = the wave-length of sodium light, then $\frac{2l}{\lambda} = 5 \cdot 10^7$ cms. On the other hand $\frac{q^2}{c^2}$ is of the order $\left(\frac{30 \text{ kilometres}}{300,000 \text{ kilometres}} \right)^2$, that is, 10^{-8} . The expected displacement of the

fringes would thus have to be about 0.56 of the breadth of a fringe. Actually, an amount of the order 0.02 of the breadth of a fringe was observed. Thus, the ether wind did not make itself remarked optically in the motion of the earth. By carrying out the experiment at different times of the year the possible objection that the motion of translation of the entire solar system might have counterbalanced the motion of the earth in her orbit was removed.

The Michelson-Morley experiment has shown conclusively that there is no physical sense in talking of absolute rest or of a translation relative to absolute space, since all systems that move rectilinearly and uniformly with respect to one another are of equal value for describing natural phenomena. It is thus a matter of convention which system we are to regard as at rest and which as being in motion. We may assign the same value to the velocity of light in all systems. A detailed theory of these fundamental experiments may be found in all comprehensive accounts of the special theory of relativity. We here merely mention the original paper by A. Einstein (*Annalen der Physik*, Bd. 17, 1905, p. 891), and the booklet, "Einführung in die Relativitätstheorie," by Dr. W. Block, out of the series "Aus Natur und Geisteswelt," Teubner, 1918.

Note 3 (p. 9). Abolishing the transformations of Newton's principle of relativity and replacing them by the so-called Lorentz-Einstein transformations signified a step of extraordinarily far-reaching consequence. It was justified in that the new theory of relativity which followed as a result of it, confirmed, without difficulty, the results of all the fundamental experiments of optics and electrodynamics. Concerning the Michelson-Morley ex-

periment, Lorentz, to account for its negative result within the realm of electrodynamics, had been compelled to set up the hypothesis that the dimensions of all bodies contract in the direction of their motion. But Einstein now showed that if we define the conception of simultaneity rigorously, taking into account the postulate of the constancy of the velocity of light, the Lorentz-transformations, which had been found empirically, followed necessarily as those equations of transformation that must hold between the co-ordinates of two systems moving uniformly and rectilinearly with respect to each other. And without the help of any further hypothesis there appears as a direct consequence of this transformation just that contraction of lengths which Lorentz had adduced to explain the result of the Michelson-Morley experiment. This contraction of a length l in the direction of motion of

an object to the value $l\sqrt{1 - \frac{v^2}{c^2}}$ is, however, in the

new theory the expression of the general fact that the dimensions of a body have only a relative meaning, that is, that their values depend on the state of motion of the observer, which determines the dimensions of the body in question. This holds for the extension of bodies in time as well as in space. From the point of view of the new principle of relativity the negative result of the Michelson-Morley experiment was self-explained. But what was the position with regard to the other fundamental facts of optics and electrodynamics? The result of Fizeau's experiment concerning the velocity of light in flowing water became a direct test of the kinematics arising out of the new formulæ. According to the Lorentz transformation the two velocities, q and v with which, for example, two locomotives approach each other, do not merely

become added, so that $q + v$ would be the relative velocity of each with respect to the other, but rather each engine-driver will find as the velocity with which he passes the other driver, the value

$$\frac{q + v}{1 + \frac{q \cdot v}{c^2}}$$

according to the new formulæ. This is the addition theorem of velocities according to the new theory. It gives us immediately the amount observed in Fizeau's experiment for the velocity of light in flowing water. Aberration and the Doppler effect follow just as readily to the correct amount. A detailed discussion of these questions is to be found in every account of the "special" theory of relativity (cf. the references given in Note 2).

Note 4 (p. 12). Ph. Frank and H. Rothe, *Ann. d. Phys.*, 4 Folge, Bd. 34, p. 825.

The assumptions for the general equations of transformation by which two systems S and S' that move *uniformly* and *rectilinearly* with the velocity q with respect to each other are connected are as follows:—

1. The equations of transformation form a linear homogeneous group in the variable parameter q . This means that the successive application of two equations of transformation, of which the one refers the system S to the system S' , and the second S' to S'' (S is to have the constant velocity q with respect to S' , and S' the constant velocity q' with respect to S'') again leads to an equation of transformation of the same form as that of the initial equations. The parameter q'' that occurs in the new equation depends in a definite way on q' and q .

2. The contractions of the lengths depend only on the value of the parameter q . We must, of

course, from the very outset reckon with the possibility that the length of a rod measured in the system that is at rest comes out differently when measured in the moving system. Now, condition 2 requires that if contractions occur (that is, changes of length in these various methods of determination) values are to depend only on the *magnitude* of the velocity of both systems and not on the direction of their motion in space. Thus this postulate endows space with the property of isotropy, and is in fair correspondence with the postulate of section 3*a*, which states that it must be possible to compare each line-element with every other in length independently of its position in space, and its direction.

An essential feature is that the constancy of the velocity of light is *not* demanded in either of the postulates 1 and 2. Rather, the distinguishing property of a *definite* velocity in virtue of which it preserves its value in *all* systems that emerge out of one another through such transformations is a direct corollary to these two general postulates, and the result of the Michelson-Morley experiment merely determines the value of this special velocity which could, of course, be found only from observation.

Note 5 (p. 15). Einstein has shown in a simple example how, on the basis of the formulæ of the special theory of relativity, a point-mass loses inertial mass when it radiates out energy.

We assume that a point-mass emits a light-wave of energy $\frac{L}{2}$ in a certain direction, and a light-wave

of the same energy $\frac{L}{2}$ in the opposite direction.

Then, in view of the symmetry of the process of emission with respect to the system of reference of the co-ordinates x, y, z, t originally chosen, the point-mass remains at rest. Let the total energy

of the point-mass be E_0 referred to this system, but H_0 referred to a second system which we suppose moving with the uniform velocity v with respect to the first. We shall apply the principle of energy to this process. If V and A are the frequency and amplitude of the light-wave in the initial system, V' , A' , x' , y' , z' , t' the frequency, amplitude, and co-ordinates in the second (the moving) system, further, ϕ the angle between the wave-normals and the line connecting the point-mass with the observer, then Doppler's principle gives for the frequency of the light-wave in the moving system :

$$\nu' = \nu \cdot \frac{1 - \frac{v}{c} \cdot \cos \phi}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

The formulæ of the special theory of relativity give us, correspondingly, for the amplitude in the moving system

$$A' = A \cdot \frac{1 - \frac{v}{c} \cos \phi}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

According to Maxwell's theory the energy of the light-wave per unit volume is $\frac{1}{8\pi} \cdot A^2$. We now wish

to calculate the corresponding energy-density also with respect to the moving system. We must here take into account that, in consequence of the contraction of the lengths according to the Lorentz-Einstein transformation formulæ, the volume V of a sphere in the resting system becomes transformed into that of an ellipsoid as measured from the moving system V' ; indeed, this volume of the ellipsoid is

$$V' = V \cdot \frac{\sqrt{1 - \frac{v^2}{c^2}}}{\sqrt{1 - \frac{v}{c} \cos \phi}}.$$

Hence the energy-densities in the accented and unaccented system are in the ratio :

$$\frac{L'}{L} = \frac{\frac{1}{8\pi} \cdot A'^2 \cdot V'}{\frac{1}{8\pi} \cdot A^2 \cdot V} = \frac{1 - \frac{v}{c} \cos \phi}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

If we now designate the energy-content of the point-mass *after* the emission by E_1 , and the corresponding quantity referred to the moving system by H_1 , then we have :

$$E_1 = E_0 - \left[\frac{L}{2} + \frac{L}{2} \right] \text{ or, } E_0 = E_1 + \left[\frac{L}{2} + \frac{L}{2} \right],$$

whereas

$$H_0 = H_1 + \left[\frac{L}{2} \frac{1 - \frac{v}{c} \cos \phi}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{L}{2} \frac{1 + \frac{v}{c} \cos \phi}{\sqrt{1 - \frac{v^2}{c^2}}} \right].$$

From this we get directly that

$$[H_0 - E_0] - [H_1 - E_1] = L \left[\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right].$$

What does this equation assert ?

H and E are the energy-values of the same point-mass, in the first place referred to a system with respect to which the point-mass moves, and in the second related to a system in which the point-mass

is at rest. Hence the difference $H - E$, except for an additive constant, must be equal to the *kinetic* energy of the point-mass referred to the moving system. Thus, we may write

$$H_0 - E_0 = K_0 + C \quad H_1 - E_1 = K_1 + C$$

wherein C denotes a constant which does not alter during the light-emission of the point-mass, since, owing to the symmetry of the process, the point-mass remains at rest with respect to the initial system. So we arrive at the relation:

$$K_0 - K_1 = L \left[\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right] = \frac{1}{2} \frac{L}{c^2} \cdot v^2 \dots$$

In words this equation states that owing to the point-mass emitting the energy L as light, its kinetic energy referred to a moving system sinks from the value K_0 to the value K_1 , corresponding to a loss in inertial mass of the amount $\frac{L}{c^2}$. For, according to classical mechanics, the expression $\frac{1}{2} m \cdot v^2$ in which m is the inertial mass of the observed body, is a measure of the kinetic energy of this body referred to a system with respect to which it moves with the velocity v . Thus $\frac{L}{c^2}$ must be taken as standing for the inertial mass of an amount of energy L .

Note 6 (p. 29). The facts that *every* pair of points (point-pair) in space have the same magnitude-relation (viz. the same expression for the mutual *distance* between them) and that with the aid of this relation, every point-pair can be compared with every other, constitute the characteristic feature which distinguishes space from the remaining

continuous manifolds which are known to us. We measure the mutual distance between two points on the floor of a room, and the mutual distance between two points which lie vertically above one another on the wall, with the same measuring-scale, which we thus apply in any direction at pleasure. This enables us to "compare" the mutual distance of a point-pair on the floor with the mutual distance of any other pair of points on the wall.

In the system of tones, on the contrary, quite different conditions prevail. The system of tones represents a manifold of two dimensions, if one distinguishes every tone from the remaining tones by its pitch and its intensity. It is, however, not possible to compare the "distance" between two tones of the *same pitch* but *different intensity* (analogous to the two points on the floor) with the "distance" between two tones of *different pitch* but *equal intensity* (analogous to the two points on the wall). The measure-conditions are thus quite different in this manifold.

In the system of colours, too, the measure-relations have their own peculiarity. The dimensions of the manifold of colours are the same as those of space, as each colour can be produced by mixing the *three* "primary" colours. But there is no relation between two arbitrary colours, which would correspond to the distance between two points in space. Only when a third colour is derived by mixing these two, does one obtain an equation between these three colours similar to that which connects three points in space lying in one straight line.

These examples, which are borrowed from Helmholtz's essays, serve to show that the measure-relations of a continuous manifold are not already given in its definition as a *continuous* manifold,

nor by fixing its dimensions. A continuous manifold generally allows of various measure-relations. It is only experience which enables us to derive the measure-laws which are valid for each particular manifold. The fact, discovered by experience, that the dimensions of bodies are independent of their particular position and motion, led to the laws of Euclidean geometry where *congruence* is the deciding factor in comparing various portions of space. These questions have been exhaustively treated by Helmholtz in various essays. References :—

Riemann, "Über die Hypothesen, welche der Geometrie zugrunde liegen" (1854). Newly published and annotated by H. Weyl, Berlin, 1919.

Helmholtz. "Ueber die tatsächlichen Grundlagen der Geometrie," *Wiss. Abh.* 2, S. 10.

Helmholtz. "Ueber die Tatsachen, welche der Geometrie zugrunde liegen," *Wiss. Abh.* 2, S. 618.

Helmholtz. "Ueber den Ursprung und die Bedeutung der geometrischen Axiome," *Vorträge und Reden*, Bd. 2, S. 1.

Note 7 (p. 26). The postulate that finite rigid bodies are to be capable of free motions, can be most strikingly illustrated in the realm of two-dimensions. Let us imagine a triangle to be drawn upon a *sphere*, and also upon a *plane*: the former being bounded by arcs of great circles and the latter by straight lines; one can then slide these triangles over their respective surfaces at will, and can make them coincide with other triangles, without thereby altering the lengths of the sides or the angles. Gauss has shown that this is possible because the *curvature* at every point of the sphere (or the plane, respectively) has exactly the same value. And yet the geometry of curves traced upon a *sphere* is different from that of curves traced

upon a *plane*, for the reason that these two configurations cannot be deformed into one another without tearing (*vide* Note 27). But upon *both* of them planimetric figures can be freely shifted about, and, therefore, theorems of congruence hold upon them. If, however, we were to define a curvilinear triangle upon an egg-shaped surface by the three shortest lines connecting three given points upon it, we should find that triangles could be constructed at different places on this surface, having the same lengths for the sides; but these sides would enclose angles different from those included by the corresponding sides of the initial triangle, and, consequently, such triangles would not be congruent, in spite of the fact that corresponding sides are equal. Figures upon an egg-shaped surface cannot, therefore, be made to slide over the surface without altering their dimensions: and in studying the geometrical conditions upon such a surface, we do not arrive at the usual theorems of congruence. Quite analogous arguments can be applied to three- and four-dimensional realms: but the latter cases *offer no corresponding pictures to the mind*. If we demand that bodies are to be freely movable in space without suffering a change of dimensions, the "*curvature*" of the space must be the same at every point. The conception of curvature, as applied to any manifold of more than two dimensions, allows of strict mathematical formulation; the term itself only hints at its analogous meaning, as compared with the conception of curvature of a surface. In three-dimensional space, too, various cases can be distinguished, similarly to plane- and spherical-geometry in two-dimensional space. Corresponding to the sphere, we have a non-Euclidean space with constant positive curvature; corresponding to the plane we

have Euclidean space with curvature zero. In both these spaces bodies can be moved about without their dimensions altering; but Euclidean space is furthermore infinitely extended: whereas "spherical" space, though unbounded, like the surface of a sphere, is not infinitely extended. These questions are to be found extensively treated in a very attractive fashion in Helmholtz's familiar essay: "Ueber den Ursprung und die Bedeutung der geometrischen Axiome" (*Vorträge und Reden*, Bd. 2, S. 1).

Note 8 (p. 26). The properties, which the analytical expression for the length of the line-element must have, may be understood from the following:

Let the numbers x_1, x_2 denote any point of any continuous two-dimensional manifold, e.g. a surface. Then, together with this point, a certain "domain" around the point is given, which includes points all of which lie in the plane.—D. Hilbert has strictly defined the conception of a multiply-extended magnitude (i.e. a manifold) upon the basis of the theory of aggregates in his "Grundlagen der Geometrie" (p. 177). In this definition the conception of the "domain" encircling a point is made to give Riemann's postulate of the *continuous* connection existing between the elements of a manifold and a strict form.

Setting out from the point x_1, x_2 we can continuously pass into its domain, and at any point, e.g. $x_1 + dx_1, x_2 + dx_2$, inquire as to the "distance" of this point from the starting-point. The function which *measures* this distance will depend upon the values of x_1, x_2, dx_1, dx_2 , and for every *intermediate* point of the path which has conducted us from x_1, x_2 to the point $x_1 + dx_1, x_2 + dx_2$ will successively assume certain continually changing, and, as we may suppose, continually *increasing*, values. At

the point x_1, x_2 itself it will assume the value zero, and for every other point of the domain its value must be positive. Moreover, we shall expect to find that, for any intermediate point, denoted by $x_1 + d\xi_1, x_2 + d\xi_2$, $d\xi_1 = \frac{1}{2}dx_1$ and $d\xi_2 = \frac{1}{2}dx_2$, the required function which measures the distance of this point from x_1, x_2 , will, at this point, have a value half that of its value for the point $x_1 + dx_1, x_2 + dx_2$. Under these assumptions, the function will be homogeneous and of the first degree in the dx 's; its value will then appear multiplied by that factor in proportion to which the dx 's were increased. In addition, it must itself vanish if all the dx 's are zero; and if they all change their sign it must not alter its value, which always remains positive. It will immediately be evident that the function

$$ds = \sqrt{g_{11}dx_1^2 + g_{12}dx_1dx_2 + g_{22}dx_2^2}$$

fulfils all these requirements; but it is by no means the only function of this kind.

Note 9 (p. 29). But the expression of the fourth degree for the line element would not permit of any geometrical interpretation of the formula, such as is possible with the expression

$$ds^2 = g_{11}dx_1^2 + g_{12}dx_1dx_2 + \dots + g_{33}dx_3^2,$$

which latter may be regarded as a general case of Pythagoras' theorem.

Note 10 (p. 30). By a "discrete" manifold we mean one in which no continuous transition of the single elements from one to another is possible, but each element to a certain extent represents an independent entity. The aggregate of all whole numbers, for instance, is a manifold of this type, or the aggregate of all *planets* in our solar system, etc., and many other examples may be found; and

indeed all finite aggregates in the theory of aggregates are such discrete manifolds. "Measuring," in the case of discrete manifolds, is performed merely by "counting," and does not present any special difficulties, as all manifolds of this type are subject to the same principle of measurement. When Riemann then proceeds to say: "Either, therefore, the reality which underlies space must form a discrete manifold, or we must seek the ground of its metric relations outside it, in binding forces which act upon it," he only wishes to hint at a possibility, which is at present still remote, but which must, in principle, always be left open. In just the last few years a similar change of view has actually occurred in the case of another manifold which plays a very important part in physics, viz. "*energy*"; the meaning of the hint Riemann gives will become clearer if we consider this example.

Up till a few years ago, the energy which a body emanates by radiation was regarded as a continuously variable quantity: and attempts were therefore made to measure its amount at any particular moment by means of a continuously varying sequence of numbers. The researches of Max Planck have, however, led to the view that this energy is emitted in "*quanta*," and that therefore the "*measuring*" of its amount is performed by counting the number of "*quanta*." The reality underlying radiant energy, according to this, is a discrete and *not* a continuous manifold. If we now suppose that the view were gradually to take root that, on the one hand, all measurements in space only have to do with distances between ether-atoms; and that, on the other hand, the distances of single ether-atoms from one another can only assume certain definite values, all distances in space would

be obtained by "counting" these values, and we should have to regard space as a discrete manifold.

Note 11 (p. 32). C. Neumann. "Ueber die Prinzipien der Galilei-Newtonschen Theorie," Leipzig 1870, S. 18.

Note 12 (p. 32). H. Streintz. "Die physikalischen Grundlagen der Mechanik," Leipzig, 1883.

Note 13 (p. 33). A. Einstein. "Annalen der Physik," 4 Folge, Bd. 17, S. 891.

Note 14 (p. 35). Minkowski was the first to call particular attention to this deduction of the special principle of relativity.

Note 15 (p. 38). The term "inertial system" was originally not associated with the system, which Neumann attached to the hypothetical body Alpha. Nowadays it is generally understood to signify a rectilinear system of co-ordinates, relatively to which a point-mass, which is only subject to its own inertia, moves uniformly in a straight line. Whereas C. Neumann only invented the body Alpha, as an absolutely hypothetical configuration, in order to be able to formulate the law of inertia, later researches, especially those of Lange, tended to show that, on the basis of rigorous kinematical considerations, a co-ordinate system could be derived, which would possess the properties of such an inertial system. However, as C. Neumann and J. Petzoldt have demonstrated, these developments contain faulty assumptions, and give the law of inertia no firmer basis than the body Alpha introduced by Neumann.

Such an inertial system is determined by the straight lines which connect three point-masses infinitely distant from one another (and thus unable to exert a mutual influence upon one another) and which are not subject to any other forces. This definition makes it evident why no inertial system

will be discoverable in nature, and why, consequently, the law of inertia will never be able to be formulated so as to satisfy the physicist. References :—

C. Neumann. "Ueber die Prinzipien der Galilei-Newtonschen Theorie," Leipzig, 1870.

L. Lange. "Berichte der Kgl. Sächs. Ges. d. Wissenschaften. Math.-phil. Klasse," 1885.

L. Lange. "Die Geschichte der Entwicklung des Bewegungsbegriffes," Leipzig, 1886.

H. Seeliger. "Ber. der Bayr. Akademie," 1906, Heft 1.

C. Neumann. "Ber der Kgl. Sächs. Ges. d. Wiss. Math.-phys. Klasse," 1910, Bd. 62, S. 69 and 383.

J. Petzoldt. "Ann. der Naturphilosophie," Bd. 7.

Note 16 (p. 38). E. Mach. "Die Mechanik in ihrer Entwicklung," 4 Aufl. S. 244.

Note 17 (p. 40). The new points of view as to the nature of inertia are based upon the study of the electromagnetic phenomena of radiation. The special theory of relativity, by stating the theorem of the inertia of energy, organically grafted these views on to the existing structure of theoretical physics. The dynamics of cavity-radiation, i.e. the dynamics of a space enclosed by walls without mass, and filled with electromagnetic radiation, taught us that a system of this kind opposes a resistance to every change of its motion, just like a heavy body in motion. The study of electrons (free electric charges) in a state of free motion, e.g. in a cathode-tube, taught us likewise that these exceedingly small particles behave like inert bodies; that their inertia is not, however, conditioned by the matter to which they might happen to be attached, but rather by the electromagnetic effects of the field to which the moving electron is subject. This gave rise to the conception of the apparent (electromagnetic) mass of an

electron. The special theory of relativity finally led to the conclusion that to *all* energy must be accorded the property of inertia.

Every body contains energy (e.g. a certain definite amount in the form of heat-radiation internally). The inertia, which the body reveals, is thus partly to be debited to the account of this contained energy. As this share of inertia is, according to the special theory of relativity, relative (i.e. represents a quantity which depends upon the choice of the system of reference), the whole amount of the inertial mass of the body has no absolute value, but only a relative one. This energy-content of radiant heat is distributed throughout the whole volume of each particular body; one can thus speak of the energy-content of unit volume. This enables us to derive the notion of density of energy. The density of the energy (i.e. amount per unit volume) is thus a quantity, the value of which is also dependent upon the system of reference. References:—

M. Planck. "Ann. der Phys.," 4 Folge, Bd. 26.

M. Abraham. "Electromagnetische Energie der Strahlung," 4 Aufl., 1908.

Note 18 (p. 40). The determination of the inertial mass of a body by measuring its weight is rendered possible only by the experimental fact that all bodies fall with equal acceleration in the gravitational field at the earth's surface. If p and p' denote the pressures of two bodies upon the same support (i.e. their respective weights), and g denote the acceleration due to the earth's gravitational field at the point in question, then $p = m \cdot g$ dynes and $p' = m' \cdot g$ dynes, respectively, where m and m' are the factors of proportionality, and are called the *masses* of the two bodies, respectively. As g has the same value in both equations, we have

$$\frac{p'}{p} = \frac{m'}{m},$$

and we can accordingly measure the *masses* of two bodies at the same place, by determining their *weights*.

Although Galilei and Newton had already known that all bodies at the same place fall with the same velocity (if the resistance of the air be eliminated), this very remarkable fact has not received any recognition in the foundations of mechanics. Einstein's principle of equivalence is the first to assign to it the position to which it is, beyond doubt, entitled.

Note 19 (p. 41). Arguing along the same lines B. and J. Friedländer have suggested an experiment to show the relativity of rotational motions, and, accordingly, the reversibility of centrifugal phenomena ("Absolute and Relative Motion," Berlin, Leonhard Simion, 1896). On account of the smallness of the effect, the experiment cannot, at present, be performed successfully; but it is quite appropriate for making the physical content of this postulate more evident. The following remarks may be quoted:

"The torsion-balance is the most sensitive of all instruments. The largest rotating-masses, with which we can experiment, are probably the large fly-wheels in rolling-mills and other big factories. The centrifugal forces assert themselves as a pressure which tends from the axis of rotation. If, therefore, we set up a torsion-balance in somewhat close proximity to one of these large fly-wheels, in such a position that the point of suspension of the movable part of the torsion-balance (the needle) lies exactly, or as nearly as possible, in the continuation of the axis of the fly-wheel, the needle should endeavour

to set itself parallel to the plane of the fly-wheel, if it is not originally so, and should register a corresponding displacement. For centrifugal force acts upon every portion of mass which does not lie exactly in the axis of rotation, in such a way as to tend to increase the distance of the mass from the axis. It is immediately apparent that the greatest possible displacement-effect is attained when the needle is parallel to the plane of the wheel."

This proposed experiment of B. and J. Friedländer is only a variation of the experiment which persuaded Newton to his view of the absolute character of rotation. Newton suspended a cylindrical vessel filled with water by a thread, and turned it about the axis defined by the thread till the thread became quite stiff. After the vessel and the contained liquid had completely come to rest, he allowed the thread to untwist itself again, whereby the vessel and the liquid started to rotate rapidly. He thereby made the following observations. Immediately after its release the *vessel alone* assumed a motion of rotation, since the friction (viscosity) of the water was not sufficient to transmit the rotation immediately to the water. So long as this state of affairs prevailed, the surface of the water remained a horizontal plane. But the more rapidly the water was carried along by the rotating walls of the vessel, the more definitely did the centrifugal forces assert themselves, and drive the water up the walls, so that finally its free surface assumed the form of a paraboloid of revolution. From these observations Newton concluded that the *rotation* of the walls of the vessel *relative* to the water does not call up forces in the latter. Only when the water itself shares in the rotation, do the centrifugal forces make their appearance. From this he came to his conclusion of the absolute character of rotations.

This experiment became a subject of frequent discussion later: and E. Mach long ago objected to Newton's deduction, and pointed out that it cannot be straightway affirmed that the rotation of the walls of the vessel relative to the water is entirely without effect upon the latter. He regards it as quite conceivable that, provided the mass of the vessel were large enough, e.g. if its walls were many kilometres thick, then the free surface of the water which is at rest in the rotating vessel would not remain plane. This objection is quite in keeping with the view entailed by the general theory of relativity. According to the latter, the centrifugal forces can also be regarded as gravitational forces, which the total sum of the masses rotating around the water exerts upon it. The gravitational effect of the walls of the vessel upon the enclosed liquid is, of course, vanishingly small compared with that of all the masses in the universe. It is only when the water is in rotation relatively to all these masses that perceptible centrifugal forces are to be expected. The experiment of B. and J. Friedländer was intended to refine the experiment performed by Newton, by using a sensitive torsion-balance susceptible to exceedingly small forces in place of the water, and by substituting a huge fly-wheel for the vessel which contained the water. But this arrangement, too, can lead to no positive result, as even the greatest fly-wheel at present available represents only a vanishingly small mass compared with the sum-total of masses in the universe.

Note 20 (p. 42). We use the term "field of force" to denote a field in which the force in question varies continuously from place to place, and is given for each point in the field by the value of some function of the place. The centrifugal forces in the

interior and on the outer surface of a rotating body are so distributed as to compose a field of this kind throughout the whole volume of the body, and there is nothing to hinder us from imagining this field to extend outwards beyond the outer surface of the body, e.g. beyond the surface of the earth into its own atmosphere. We can thus briefly speak of the whole field as the centrifugal field of the earth ; and, as the centrifugal field, according to the older views, is conditioned only by the inertia of bodies, and not by their gravitation, we can further speak of it as an inertial field, in contra-distinction to the gravitational field, under the influence of which all bodies which are not suspended or supported fall to earth.

Accordingly the effects of various fields of force are superposed at the earth's surface : (1) the effect of the gravitational field, due to the gravitation of the particles of the earth's mass towards one another, and which is directed towards the centre of the earth ; (2) the effect of the centrifugal field, which, according to Einstein's view, can be regarded as a gravitational field, and the direction of action of which is outwards and parallel to the plane of the meridian of latitude ; finally (3) the effect of the gravitational field, due to the various heavenly bodies, foremost amongst them, the sun and the moon.

Note 21 (p. 42). Eötvös has published the results of his measurements in the " *Mathematische und Naturwissenschaftliche Berichte aus Ungarn*," Bd. 8, S. 64, 1891. A detailed account is given by D. Pekár, " *Das Gesetz der Proportionalität von Trägheit und Gravitation*." " *Die Naturwissenschaft*," 1919, 7, p. 327.

Whereas the earlier investigations of Newton and Bessel (" *Astr. Nachr.*" 10, S. 97, and " *Abhandlungen von Bessel*," Bd. 3, S. 217), about the

attractive effect of the earth upon various substances, are based upon observations with a pendulum, Eötvös worked with sensitive torsion-balances.

The force, in consequence of which bodies fall, is composed of two components: first the attractive force of the earth, which (except for deviations which may, for the present, be neglected) is directed towards the centre of the earth; and, second, the centrifugal force, which is directed outwards parallel to the meridians of latitude. If the attraction of the earth upon two bodies of equal mass but of different substance were different, the resultant of the attractive and centrifugal forces would point in a different direction for each body. Eötvös then states: "By calculation we find that if the attractive effect of the earth upon two bodies of equal mass, but composed of different substance, differed by a thousandth, the directions of the gravitational forces acting upon the two bodies respectively would make an angle of 0.356 second, i.e. about a third of a second with one another;" and if the difference in the attractive force were to amount to a twenty-millionth, this angle would have to be $\frac{356}{20,000,000}$ th seconds; that is, slightly more than $\frac{1}{60,000}$ th of a second; and later:

"I attached separate bodies of about 30 grms. weight to the end of a balance-beam about 25 to 30 cms. long, suspended by a thin platinum wire in my torsion-balance. After the beam had been placed in a position perpendicular to the meridian, I determined its position exactly by means of two mirrors, one fixed to it and another fastened to the case of the instrument. I then turned the instrument, together with the case, through 180°, so that the body which was originally at the east end of the beam now arrived at the west end: I then determined the position of the beam again, relative

to the instrument. If the resultant weights of the bodies attached to both sides pointed in different directions, a torsion of the suspending wire should ensue. But this did not occur in the cases in which a brass sphere was constantly attached to the one side, and glass, cork, or crystal antimony was attached to the other; and yet a deviation of $\frac{1}{60,000}$ th of a second in the direction of the gravitational force would have produced a torsion of one minute, and this would have been observed accurately."

Eötvös thus attained a degree of accuracy, such as is approximately reached in weighing; and this was his aim: for his method of determining the mass of bodies by weighing is founded upon the axiom that the attraction exerted by the earth upon various bodies depends only upon their mass, and not upon the substance composing them. This axiom had, therefore, to be verified with the same degree of accuracy as is attained in weighing. If a difference of this kind in the gravitation of various bodies having the same mass but being composed of different substance exists at all, it is, according to Eötvös, less than a twenty-millionth for brass, glass, antimonite, cork, and less than a hundred-thousandth for air.

Note 22 (p. 44). *Vide* also A. Einstein, "Grundlagen der allgemeinen Relativitätstheorie," "Ann. d. Phys.," 4 Folge, Bd. 49, S. 769.

Note 23 (p. 46). The equation $\delta\{ds\} = 0$ asserts that the variation in the length of path between two sufficiently near points of the path vanishes for the path actually traversed; i.e. the path actually chosen between two such points is the shortest of all possible ones. If we retain the view of classical mechanics for a moment, the following example will give us the sense of the principle

clearly: In the case of the motion of a point-mass, free to move about in space, the straight line is always the shortest connecting line between two points in space: and the point-mass will move from the one point to the other along this straight line, provided no other disturbing influences come into play (Law of inertia). If the point-mass is constrained to move over any curved surface, it will pass from one point to another along a geodetic line to the surface, since the geodetic lines represent the shortest connecting lines between points on the surface. In Einstein's theory there is a fully corresponding principle, but of a much more general form. Under the influence of *inertia and gravitation* every point-mass passes along the geodetic lines of the space-time-manifold. The fact of these lines not, in general, being straight lines, is due to the gravitational field, in a certain sense, putting the point-mass under a sort of constraint, similar to that imposed upon the freedom of motion of the point-mass by a curved surface. A principle in every way corresponding had already been installed in mechanics as a fundamental principle for all motions by Heinrich Hertz.

Note 24 (p. 48). *Vide* A. Einstein, "Ann. d. Phys.," 4 Folge, Bd. 35, S. 898.

Note 25 (p. 48). The expression "acceleration-transformation" means that the equations giving the transformation from the variables x, y, z, t to the system of variables x_1, x_2, x_3, x_4 , which is the basis of our discussion, can be regarded as giving the relations between two systems of reference which are moving with an *accelerated* motion relatively to one another. The nature of the state of motion of two systems of reference relative to one another finds its expression in the analytical form of the equations of transformation of their coordinates.

Note 26 (p. 51). Two things are to be undertaken in the following: (1) the fundamental equations of the new theory are to be written in an explicit form, and (2) the transition to Newton's fundamental equations is to be performed.

1. From the equation of variation $\delta\{ds\} = 0$ where

$$ds^2 = \sum_{\mu, \nu}^4 g_{\mu\nu} dx_{\mu} dx_{\nu},$$

we have, after carrying out the operation of variation, the four total differential equations :

$$(I) \quad \frac{d^2 x_\sigma}{ds^2} = \sum_{\mu\nu} \Gamma_{\mu\nu}^\sigma \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} \quad (\sigma = 1, 2, 3, 4).$$

These are the equations of motion of a material point in the gravitational field defined by the $g_{\mu\nu}$'s.

The symbol $\Gamma_{\mu\nu}^{\sigma}$ here denotes the expression

$$-\frac{1}{2} \sum_a g^{\sigma a} \left(\frac{\delta g_{\mu a}}{\delta x_\nu} + \frac{\delta g_{\nu a}}{\delta x_\mu} - \frac{\delta g_{\mu\nu}}{\delta x_a} \right).$$

The symbol $\overline{g^{\sigma a}}$ denotes the minor of $g_{\sigma a}$ in the determinant

[illegible]

divided by the determinant itself.

The ten differential equations for the "gravitational potentials" $g_{\mu\nu}$ are:

$$(2) \quad \frac{d\Gamma_a}{dx_a} + \sum_{\alpha\beta} \Gamma_{\mu\beta}^\alpha \Gamma_{\nu\alpha}^\beta = \kappa(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T).$$

The quantities $T_{\mu\nu}$ and T are expressions which are related in a simple manner to the components of the stress-energy-tensor (which plays the part

of the quantity exciting the field in the new theory in place of the density of mass). κ is essentially equal to the gravitational constant of Newton's theory.

The differential equations (1) and (2) are the fundamental equations of the new theory. The derivation of these equations is carried out in detail in the tract by A. Einstein, "The Foundations of the General Principle of Relativity," J. A. Barth, Leipzig, 1916.

2. In order to obtain a connection between these equations and Newton's theory, we must make several simplifying assumptions. We shall first assume that the $g_{\mu\nu}$'s differ only by quantities which are small compared with unity from the values given by the scheme :

$$\begin{Bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{Bmatrix} = \begin{Bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{Bmatrix}.$$

These values for the $g_{\mu\nu}$'s characterize the case of the special theory of relativity, i.e. the case of the condition free of gravitation. We shall also assume that, at infinite distances, the $g_{\mu\nu}$'s tend to, and do finally, assume the above values; that is, that matter does not extend into infinite space.

Secondly, we shall assume that the velocities of matter are small compared with the velocity of light, and can be regarded as small quantities of the first order. The quantities

$$\frac{dx_1}{ds}, \quad \frac{dx_2}{ds}, \quad \frac{dx_3}{ds},$$

will then be infinitely small quantities of the first order, and $\frac{dx_4}{ds}$ will equal 1, except for quantities of

the second order. From the equations defining the $\Gamma_{\mu\nu}^\sigma$ it will then be seen that these quantities will be infinitely small, of the first order. If we neglect quantities of the second order, and finally assume that, for small velocities of matter, the changes of the gravitational field with respect to time are small (i.e. that the derivatives of the $g_{\mu\nu}$'s with respect to time may be neglected in comparison with the derivatives taken with regard to the space-co-ordinates) the system of equations (I) assumes the form :

$$(Ia) \quad \frac{d^2 x_\tau}{dt^2} = - \frac{1}{2} \frac{\delta g_{44}}{\delta x_\tau} (\tau = 1, 2, 3).$$

This would be the equation of motion of a point-mass as already given by Newton's mechanics, if $\frac{1}{2}g_{44}$ be taken as representing the ordinary gravitational potential. It still remains to be seen what the differential equation for g_{44} becomes in the new theory under the simplifying assumptions we have chosen.

The stress-energy tensor, which excites the field, degenerates, as a result of our quite special assumptions, into the density of mass ρ :

$$T = T_{44} = \rho.$$

In the differential equations (2) the second term on the left-hand side is the product of two magnitudes, which, according to the above arguments, are to be regarded as infinitely small quantities of the first order. Thus the second term, being of the second order of small quantities, may be dismissed. The first term, on the other hand, if we omit the terms differentiated with respect to time, as above (i.e. if we regard the gravitational field as "stationary"), reduces to :

$$- \frac{1}{2} \left(\frac{\delta^2 g_{44}}{\delta x_1^2} + \frac{\delta^2 g_{44}}{\delta x_2^2} + \frac{\delta^2 g_{44}}{\delta x_3^2} \right) = - \frac{1}{2} \Delta g_{44} \text{ for } \mu = \nu = 4.$$

The differential equation for g_{44} thus degenerates into Poisson's equation :

$$(2a) \quad \Delta g_{44} = \kappa \rho.$$

Thus, to a first approximation (i.e. if one regards the velocity of light as infinitely great, and this is a characteristic feature of the classical theory, as was explained in detail in § 3 (b) : if certain simple assumptions are made about the behaviour of the $g_{\mu\nu}$'s at infinity ; and if the time-changes of the gravitational field are neglected) the well-known equations of Newtonian mechanics emerge out of the differential equations of Einstein's theory, which were obtained from perfectly general beginnings.

Note 27 (p. 53). The theory of surfaces, i.e. the study of geometry upon surfaces, makes it immediately apparent that the theorems, which have been established for any surface, also hold for any surface which can be generated by distorting the first *without tearing*. For if two surfaces have a point-to-point correspondence, such that the line-elements are equal *at corresponding points*, then corresponding finite arcs, angles, and areas, etc., will be equal. One thus arrives at the same planimetric theorems for the two surfaces. Such surfaces are called "*deformable*" surfaces. The necessary and sufficient condition that surfaces be continuously deformable is that the expression for the line-element of the one surface

$$ds^2 = g_{11}dx_1^2 + g_{12}dx_1dx_2 + g_{22}dx_2^2$$

can be transformed into that for the other,

$$ds'^2 = g_{11}'dx_1'^2 + g_{12}'dx_1'dx_2' + g_{22}'dx_2'^2.$$

According to Gauss, it is *necessary* that both surfaces have equal *measures of curvature*. If the latter

is constant over the whole surface, as e.g. in the case of a cylinder or a plane, all conditions for the deformability of the surfaces are fulfilled. In other cases, special equations offer a criterion as to whether surfaces, or portions of surfaces, are deformable into one another. The numerous subsidiary problems, which result out of these questions, are discussed at length in every book dealing with differential geometry (e.g. Bianchi-Lukat).^{*} This branch of training, which was hitherto of interest only to mathematicians, now assumes very considerable importance for the physicist too.

Note 28 (p. 61). One must avoid being deceived into the belief that Newton's fundamental law is in any way to be regarded as an *explanation* of gravitation. The conception of attractive force is borrowed from our muscular sensations, and has therefore no meaning when applied to dead matter. C. Neumann, who took great pains to place Newton's mechanics on a solid basis, glosses upon this point himself in a drastic fashion, in the following narrative, which shows up the weaknesses of the former view :

"Let us suppose an explorer to narrate to us his experiences in yonder mysterious ocean. He had succeeded in gaining access to it, and a remarkable sight had greeted his eyes. In the middle of the sea he had observed two floating icebergs, a larger and a smaller one, at a considerable distance from one another. Out of the interior of the larger one, a voice had resounded, issuing the following command in a peremptory tone: 'Ten feet nearer!' The little iceberg had immediately carried out the order, approaching ten feet nearer the larger one. Again, the larger gave out the order: 'Six feet nearer!' The other had again immediately

^{*} Forsyth's "Differential Geometry."—H. L. B.

executed it. And in this manner order after order had echoed out: and the little iceberg had continually been in motion, eager to put every command immediately and implicitly into action.

"We should certainly consign such a report to the realm of fables. But let us not scoff too soon! The ideas, which appear so extraordinary to us in this case, are exactly the same as those which lie at the base of the most complete branch of natural science, and to which the most famous of physicists owes the glory attached to his name.

"For in cosmic space such commands are continually resounding, proceeding from each of the heavenly bodies—from the sun, planets, moons, and comets. Every single body in space hearkens to the orders which the other bodies give it, always striving to carry them out punctiliously. Our earth would dash through space in a straight line, if she were not controlled and guided by the voice of command, issuing from moment to moment, from the sun, in which the instructions of the remaining cosmic bodies are less audibly mingled.

"These commands are certainly given just as *silently* as they are obeyed; and Newton has denominated this play of interchange between commanding and obeying by another name. He talks quite briefly of a mutual attractive force, which exists between cosmic bodies. But the fact remains the same. For this mutual influence consists in one body dealing out orders, and the other obeying them."

ON THE THEORY OF RELATIVITY

By Henry L. Brose, M.A.

INTRODUCTORY

PHYSICS, being a science of observation which seeks to arrange natural phenomena into a consistent scheme by using the methods and language of mathematics, has to inquire whether the assumptions implied in any branch of mathematics used for this purpose are legitimate in its sphere, or whether they are merely the outcome of convention, or have been built up from abstract notions containing foreign elements. The use of a unit length as an unalterable measure, or of a time-division, has been accepted in traditional mechanics without inconsistency manifesting itself in general until the field of electrodynamics became accessible to investigators and rendered a re-examination of the foundations of our modes of measurement necessary. It is upon these that the whole science of mathematical physics rests. The road of advance of all science is in like manner conditioned by the inter-play of observations and notions, each assisting the other in giving us a clearer view of Nature regarded purely as a physical reality. The discovery of additional phenomena presages a still greater unification, revealing new relations and exposing new differences ; the ultimate aim of physics would seem to consist in reaching

perfect separation and distinctness of detail simultaneously with perfect co-ordination of the whole. "The all-embracing harmony of the world is the true source of beauty and is the real truth," as Poincaré has expressed it. The noblest task of co-ordinating *all* knowledge falls to the lot of philosophy.

A principle which has proved fruitful in one sphere of physics suggests that its range may be extended into others ; nowhere has this led to more successful results than in the increasing generalization which has characterized the advance of the principle of relativity. This advance is marked by three stages, quite distinct, indeed, in the nucleus of their growth, yet each succeeding stage including the *results* of the earlier.

Relativity first makes its appearance as a governing principle in Newtonian or Galilean mechanics ; difficulties arising out of the study of the phenomena of radiations led to a new enunciation of the principle upon another basis by Einstein in 1905, an enunciation which comprised the phenomena of both mechanics and radiation : this will be referred to as the "special" principle of relativity to distinguish it from the "general" principle of relativity enunciated by Einstein in 1915, and which applied to all physical phenomena and every kind of motion. The latter theory also led to a new theory of gravitation.

I. THE MECHANICAL THEORY OF RELATIVITY

In order to arrive at the precise significance of the principle of relativity in the form in which it held sway in classical mechanics, we must briefly analyse the terms which will be used to express it.

Mechanics is usually defined as the science which describes how the "position of bodies in 'space' alters with the 'time.'" We shall for the present discuss only the term "position," which also involves "distance," leaving time and space to be dealt with later when we have to consider the meaning of physical simultaneity. Modern pure geometry starts out from certain conceptions such as "point," "straight line," and "plane," which were originally abstracted from natural objects and which are *implicitly* defined by a number of irreducible and independent axioms; from these a series of propositions is deduced by the application of logical rules which we feel compelled to regard as legitimate. The great similarity which exists between geometrically constructed figures and objects in Nature has led people erroneously to regard these propositions as true: but the truth of the propositions depends on the truth of the axioms from which the propositions were logically derived. Now empirical truth implies *exact* correspondence with reality. But *pure* geometry by the very nature of its genesis excludes the test of truth. There are no geometrical points or straight lines in Nature, nor geometrical surfaces; we only find coarse approximations which are helpful in representing these mathematically conceived elements.

If, however, certain principles of mechanics are conjoined with the axioms of geometry, we leave the realm of pure geometry and obtain a set of propositions which may be verified by comparison with experience, but only within limits, viz. in respect to numerical relations, for again no exact correspondence is possible, merely a superposition of geometrical points with places occupied by matter. Our idea of the form of space is derived from the behaviour of matter, which, indeed, conditions it.

Space itself is amorphous, and we are at liberty to build up any geometry we choose for the purpose of making empirical content fit into it. Neither Euclidean, nor any of the forms of meta-geometry, has any claim to precedence. We may select for a consistent description of physical phenomena whichever is the more convenient, and requires a minimum of auxiliary hypotheses to express the laws of physical nature.

Applied geometry is thus to be treated as a branch of physics. We are accustomed to associate two points on a straight line with two marks on a (practically) rigid body: when once we have chosen an arbitrary, rigid body of reference, we can discuss motions or events mechanically by using the body as the seat of a set of axes of co-ordinates. The use of the rule and compasses gives us a physical interpretation of the distance between points, and enables us to state this distance by measurement numerically, inasmuch as we may fix upon an arbitrary unit of length and count how often it has to be applied end to end to occupy the distance between the points. Every description in space of the scene of an event or of the position of a body consists in designating a point or points on a rigid body imagined for the purpose, which coincides with the spot at which the event takes place or the object is situated. We ordinarily choose as our rigid body a portion of the earth or a set of axes attached to it.

Now Newton's (or Galilei's) law of motion states that a body which is sufficiently far removed from all other bodies continues in its state of rest or uniform motion in a straight line. This holds very approximately for the fixed stars. If, however, we refer the motion of the stars to a set of axes fixed to the earth, the stars describe circles of immense

radius; that is, for such a system of reference the law of inertia only holds approximately. Hence we are led to the definition of Galilean systems of co-ordinates. **A Galilean system is one, the state of motion of which is such that the law of inertia holds for it.** It follows naturally that Newtonian or Galilean mechanics is valid only for such Galilean or inertial systems of co-ordinates, i.e. in formulating expressions for the motion of bodies we must choose some such system at an immense distance where the Newtonian law would hold. It will be noticed that this is an abstraction, and that such a system is merely postulated by the law of motion. It is the foundation of classical mechanics, and hence also of the first or "mechanical" principle of relativity.

If we suppose a crow flying in a straight line at uniform velocity with respect to the earth diagonally over a train likewise moving uniformly and rectilinearly with respect to the earth (since motion is change of position we must specify our rigid body of reference, viz. the earth), then an observer in the train would also see the crow flying in a straight line, but with a different uniform velocity, judged from a system of co-ordinates attached to the train. We may consider both the train and the earth to be carriers of inertial systems as we are only dealing with small distances. We can then formulate the mechanical principle of relativity as follows:—

If a body be moving uniformly and rectilinearly with respect to a co-ordinate system K then it will likewise move uniformly and rectilinearly with respect to a second co-ordinate system K' , provided that the latter be moving uniformly and rectilinearly with respect to the first system K .

In our illustration, the crow represents the body, K is the earth, and K' is carried by the train.

Or, we may say that if K be an inertial system then K' , which moves uniformly and rectilinearly with respect to K , is also an inertial system. Hence, since the laws of Newtonian mechanics are based on inertial systems, it follows that all such systems are equivalent for the description of the laws of mechanics: no one system amongst them is unique, and we cannot define absolute motion or rest; any systems moving with mutual rectilinear uniform motion may be regarded as being at rest. Mathematically, this means that **the laws of mechanics remain unchanged in form for any transformation from one set of inertial axes to another.**

The development of electrodynamics and the phenomena of radiation generally showed, however, that the laws of radiation in one inertial system did not preserve their form when referred to another inertial system: K and K' were no longer equivalent for the description of phenomena such as that of light passing through a moving medium. This meant that either there was a unique inertial system enabling us to define absolute motion and rest in nature, or that we would have to build up a theory of relativity, *not* on the inertial law and inertial systems, but on some new foundation which would definitely ensure that the form of *all* physical laws would be preserved in passing from one system of reference to another.

This dilemma arose out of the conflicting results of two experiments, viz. Fizeau's and Michelson and Morley's.

Fizeau's experiment was designed to determine whether the velocity of light through moving liquid media was different from that through a stationary medium, i.e. whether the motion of the liquid caused a drag on the æther, which it would do if the mechanical law of relativity held for light

phenomena, for then the light ray would be in the same position as a swimmer travelling upstream or downstream respectively.*

No "ether-drag" was, however, detected; only a fraction of the velocity of the liquid seemed to be added to the velocity of light (c) under ordinary conditions, and this fraction depends on the refractive index of the liquid, and had previously been calculated by Fresnel: for a vacuum this fraction vanishes.

This result seemed to favour the hypothesis of a fixed ether, as was supported by Fresnel and Lorentz. But a fixed ether implies that we should be able to detect absolute motion, that is, motion with respect to the ether.

Arguing from this, let us consider an observer in the liquid moving with it. *If* there is a fixed ether, he should find a lesser value for the velocity of light (i.e. $< c$) owing to his own velocity in the same direction, or vice versa in the opposite direction.

But we on the earth are in the position of the observer in the liquid since we revolve around the sun at the rate of, approximately, 30 kms. per second (i.e. $\frac{c}{10,000}$), and we are subject to a translatory motion of about the same magnitude: hence we should be able to detect a change in the velocity of light due to our change of motion through the ether. These considerations give rise to Michelson and Morley's experiment.

Michelson and Morley attempted to detect motion relative to the supposedly fixed ether by the interference of two rays of light, one travelling in the

* It is well known that it takes a swimmer longer to travel a certain distance up and down stream than to swim across the stream and back an equal distance.

direction of motion of the earth's velocity, the other travelling across this direction of motion.

No change in the initial interference bands was, however, observed when the position of the instrument was changed, although such an effect was easily within the limits of accuracy of the experiment. Many modifications of the experiment likewise failed to demonstrate the presence of an "ether-wind."

To account for these negative results as contradicting deductions from Fizeau's experiment, Fitzgerald and, later, independently, Lorentz suggested the theory that bodies automatically contract when moving through the ether, and since our measuring scales contract in the same ratio, we are unable to detect this alteration in length; this effect would lead us always to get the same result for the velocity of light. This contraction-hypothesis agrees well with the electrical theory of matter and may be attributed to changes in the electromagnetic forces, acting between particles, which determine the equilibrium of a so-called rigid body.

Thus Michelson and Morley's experiment seems to prove that the principle of relativity of mechanics also holds for radiation effects, that is, it is impossible to determine absolute motion through the ether or space: this implies that there is no unique system of co-ordinates. It disagrees with Fizeau's result and seems to indicate the existence of a "moving ether," i.e. an ether which is carried along by moving bodies, as was upheld by Stokes and Hertz. Lord Rayleigh pointed out that if the contraction-hypothesis of Lorentz and Fitzgerald were true, isotropic bodies ought to become anisotropic on account of the motion of the earth, and that consequently, phenomena of double refraction should make their appearance. Experiments which he

himself conducted with carbon bisulphide and others carried out by Bruce with water and glass produced a negative result.

II. THE "SPECIAL" THEORY OF RELATIVITY

Einstein, in the special theory of relativity, surmounts these difficulties by doing away with the ether (as a substance) and assumes that light-signals project themselves *as such* through space. Faraday had already long ago expressed the opinion that the field in which radiations take place must not be founded upon considerations of matter, but rather that matter should be regarded as singularities or places of a singular character in the field. We may retain the name "ether" for the field as long as we do not regard it as composed of matter of the kind we know. Einstein arrives at these conclusions by critically examining our notions of space and time or of distance and simultaneity.

We know what simultaneity (time-coincidence of two events) means for our consciousness, but in making use of the idea of simultaneity in physics, we must be able to prove by actual experiment or observation that two events are simultaneous according to some definition of simultaneity. **A conception only has meaning for the physicist if the possibility of verifying that it agrees with actual experience is given.** In other words, we must have a definition of simultaneity which gives us an immediate means of proving by experiment whether, e.g. two lightning-strokes at different places occur simultaneously for an observer situated somewhere between them or not. Whenever measurements are undertaken in physics two points are made to coincide, whether they be marks on a scale and on an object, or whether they be cross-wires in a tele-

scope which have been made to coincide with a distant object to allow angular measurements to be made ; *coincidence is the only exact mode of observation, and lies at the bottom of all physical measurements.* The same importance attaches to simultaneity, which is coincidence in time. It is to be noted that no definition will be made for simultaneity occurring at (practically) one point : for this case psychological simultaneity must be accepted as the basis : the necessity for a physical definition arises only when two events happening at great distances apart are to be compared as regards the moment of their happening. We cannot do more than reduce the simultaneity of two events happening a great distance apart to simultaneity referred to a single observer at one point : this would satisfy the requirements of physics.

Einstein, accepting Michelson and Morley's result, introduced the convention in 1905 that light is propagated with a constant velocity ($= c$, i.e. 300,000 kms. per sec. approximately) in vacuo in all directions, and he then makes use of light-signals to connect up two events in time.

He illustrates his line of argument roughly by assuming two points, A and B, very far apart on a railway embankment and an observer at M midway between A and B, provided with a contrivance such as two mirrors inclined at 90° and adjusted so that light from A and B would be reflected into his vertical line of sight (Fig. 3).

Two events such as lightning-strokes are then to be defined as simultaneous for the observer at M if rays of light from them reach the observer at the same moment (psychologically) : i.e. if he sees the strokes in his mirror-contrivance simultaneously.

Next suppose that a very long train is moving with very great uniform velocity along the embank-

ment, and that the lightning-strokes pass through the two corresponding points A^1 and B^1 of the train thus :

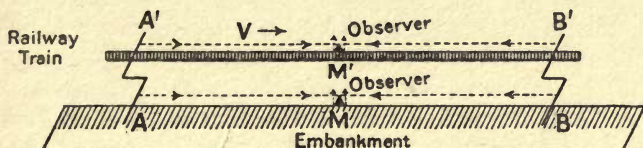


FIG. 3.

The question now arises : Are the two lightning-strokes at A and B, which are simultaneous with respect to the embankment also simultaneous with respect to the moving train ? It is quite clear that as M^1 is moving towards B^1 and away from A^1 , the observer at M^1 (mid-point of A^1B^1) will receive the ray emitted from B^1 sooner than that emitted from A^1 and he would say that the lightning-stroke at B or B^1 occurred earlier than the one at A or A^1 . Hence our condition of simultaneity is not satisfied and we are forced to the conclusion that events which are simultaneous for one rigid body of reference (the embankment) are not simultaneous for another body of reference (the train) which is in motion with regard to the first rigid body of reference. This establishes the relativity of simultaneity.

This is, of course, only an elementary example of a very special case of the regulation of clocks by light-signals. It may be asked how the mid-point M is found : one might simply fix mirrors at A and B and by flashing light-signals from points between A and B ascertain by trial the point (M) at which the return-flashes are observed simultaneously : this makes M the mid-point between the "time"-distance from A and B on the embankment.

The relativity of simultaneity states that every rigid body of reference (co-ordinate system) has its own time : a time-datum only has meaning when the

body of reference is specified, or we may say that simultaneity is dependent on the state of motion of the body of reference.

Similar reasoning applies in the case of the distance between two points on a rigid body. The length of a rod is defined as the distance, measured by (say) a metre-rule, between the two points which are occupied *simultaneously* by the two ends. Since simultaneity, as we have just seen, is relative, the distance between two points, since they depend on a simultaneous reading of two events, is also relative, and length only has a meaning if the body of reference is likewise specified: any change of motion entails a corresponding change of length: we cannot detect the change since our measures alter in the same ratio. **Length is thus a relative conception, and only reveals a relation between the observer and an object:** the "actual" length of a body in the sense we usually understand it does not exist: there is no meaning in the term. The length of a body measured parallel to its direction of motion will always yield a greater result when judged from a system attached to it than from any other system. These few remarks may suffice to indicate the relativity of distance.

In classical mechanics it had always been assumed that the time which elapsed between the happening of two events, and also the distance between two points of a rigid body were independent of the state of motion of the body of reference: these hypotheses must, as a result of the relativity of simultaneity and distance, be rejected. We may now ask whether a mathematical relation between the place and time of occurrence of various events is possible, such that every ray of light travels with the same constant velocity c whichever rigid body of reference be chosen, e.g. such that the rays measured by an

observer either in the train or on the embankment travel with the same apparent velocity.

In other words, if we assume the constancy of propagation of light in vacuo for two systems, K and K^1 moving uniformly and rectilinearly with respect to one another, what are the values of the co-ordinates x^1, y^1, z^1, t^1 of an event with respect to K^1 , if the values x, y, z, t of the same event with respect to K are given?

It is easy to arrive at this so-called Lorentz-Einstein transformation, e.g. in the case where K^1 is moving relative to K parallel to K's x axis with uniform velocity v we get:

$$x^1 = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad y^1 = y \quad z^1 = z.$$

$$t^1 = \frac{t - v/c^2 x}{\sqrt{1 - v^2/c^2}}.$$

If we put $x = ct$, we find that $\frac{x^1}{t^1}$ reduces to c .

i.e. $c = \frac{x}{t} = \frac{x^1}{t^1}$ is the same for both systems, and the condition of the constancy of c , the velocity of light in vacuo, is preserved.

If $\sqrt{1 - v^2/c^2}$ is to be real, then v cannot be greater than c , i.e. c is the limiting or maximum velocity in nature and has thus a universal significance.

If we imagine c to be infinitely great in comparison with v (and this will be the case for all ordinary velocities, such as those which occur in mechanics), the equations of transformation degenerate into:

$$x^1 = x - vt \quad y^1 = y \quad z^1 = z \quad t^1 = t.$$

This is the familiar Galilean transformation which holds for the "mechanical" principle of relativity.

We see that the Lorentz-Einstein transformation covers both mechanical and radiational phenomena.

The special theory of relativity may now be enunciated as follows: **All systems of reference which are in uniform rectilinear motion with regard to one another can be used for the description of physical events with equal justification.** That is, if physical laws assume a particularly simple form when referred to any particular system of reference, they will preserve this form when they are transformed to any other co-ordinate system which is in uniform rectilinear motion relatively to the first system. The mathematical significance of the Lorentz-Einstein equations of transformation is that the expression for the infinitesimal length of arc ds

$$(\text{viz. } ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2)$$

in the space-time* manifold x, y, z, t , preserves its form for all systems moving uniformly and rectilinearly with respect to one another.

Interpreted geometrically this means that the transformation is conformal in imaginary space of four dimensions. Moreover, the time-co-ordinate enters into physical laws in exactly the same way as the three space-co-ordinates, i.e. we may regard time spatially as a fourth dimension of space. This has been very beautifully worked out by Minkowski, whose premature loss is deeply to be regretted. It may be fitting here to recall some remarks of Bergson in his "Time and Free Will." He there states that "time is the medium in which conscious states form discrete series: this time is nothing but space, and pure duration is something different." Again,

* A continuous manifold may be defined as any continuum of elements such that a single element is defined by n continuously variable magnitudes.

“ what we call measuring time is nothing but counting simultaneities ; owing to the fact that our consciousness has organized the oscillations of a pendulum as a whole in memory, they are first preserved and afterwards disposed in a series : in a word, we create for them a fourth dimension of space, which we call homogeneous time, and which enables the movement of the pendulum, although taking place at one spot, to be continually set in juxtaposition to itself. Duration thus assumes the illusory form of a homogeneous medium and the connecting link between these two terms, space and duration, is simultaneity, which might be defined as the intersection of time and space.” Minkowski calls the space-time-manifold “ world ” and each point (event) “ world-point.”

The results achieved by the special theory of relativity may be tabulated as follows :—

- (1) It gives a consistent explanation of Fizeau's and Michelson and Morley's experiment.
- (2) It leads mathematically at once to the value suggested by Fresnel and experimentally verified by Fizeau for the velocity of a beam of light through a moving refracting medium without making any hypothesis about the physical nature of the liquid.
- (3) It gives the contraction in the direction of motion for electrons moving with high speed, without requiring any artificial hypothesis such as that of Lorentz and Fitzgerald to explain it.
- (4) It satisfactorily explains aberration, i.e. the influence of the relative motion of the earth to the fixed stars upon the direction of motion of the light which reaches us.
- (5) It accounts for the influence of the radial component of the motion of the stars, as

shown by a slight displacement of the spectral lines of the light which reaches us from the stars when compared with the position of the same lines as produced by an earth source.

- (6) It accounts for the "fine structure" of the spectral lines emitted by the atom.*
- (7) It gives the expression for the increase of inertia, owing to the addition of (apparent) electromagnetic inertia of a charged body in motion.

The last result, however, introduces an anomaly inasmuch as the inertial mass of a quickly-moving body increases, but not the gravitational mass, i.e. there is an increase of inertia without a corresponding increase of weight asserting itself. One of the most firmly established facts in all physics is hereby transgressed. This result of the theory suggested a new basis for a more general theory of relativity, viz. that proposed by Einstein in 1915. As the special theory of relativity deals only with uniform, rectilinear motions, its structure is not affected by any alteration of the ideas underlying gravitation.

III. THE GENERAL THEORY OF RELATIVITY

We have seen that the first or "mechanical" theory of relativity was built up on the notion of inertial systems as deduced from the law of inertia; the "special" theory of relativity was built up on the universal significance and invariance of c , the velocity of light in vacuo; the third or general form of relativity is to be founded on the principle of the equality of inertial and gravitational mass and in contradistinction to the other two is to hold not only for systems moving uniformly and rectilinearly with respect to one another, but for all systems whatever

* See Sommerfeld, "Atomic Structure and Spectral Lines," p. 474.

their motion ; i.e. physical laws are to preserve their form for any arbitrary transformation of the variables from one system to another.

Mass enters into the formulæ of the older physics in two forms : (1) Force = inertial mass multiplied by the acceleration. (2) Force = gravitational mass multiplied by the intensity of the field of gravitation ; or,

$$p = m \cdot a \quad p = m^1 \cdot g$$

i.e.
$$a = \frac{m^1}{m} g.$$

Observation tells us that for a given field of gravitation the acceleration is independent of the nature and state of a body ; this means that the proportionality between the two characteristic masses (inertial and gravitational) must be the same for all bodies. By a suitable choice of units we can make the factor of proportionality unity, i.e. $m = m^1$.

This fact had been noticed in classical mechanics, but not interpreted.

Eötvös in 1891 devised an experiment to test the law of the equality of inertial and gravitational mass : he argued that if the centre of inertia of a heterogeneous body did not coincide with the centre of gravity of the same body, the centrifugal forces acting on the body due to the earth's rotation acting at the centre of inertia would not, when combined with the gravitational forces acting at the centre of gravitational mass, resolve into a single resultant, but that a torque or turning couple would exist which would manifest itself, if the body were suspended by a very delicate torsionless thread or filament. His experiment disclosed that the law of proportionality of inertial and gravitational mass is obeyed with extreme accuracy : fluctuations in the ratio could only be less than a twenty-millionth.

Einstein hence assumes the exact validity of the law, and asserts that **inertia and gravitation are merely manifestations of the same quality of a body according to circumstances.** As an illustration of the purport of this equivalence he takes the case of an observer enclosed in a box in free space (i.e. gravitation is absent) to the top of which a hook is fastened. Some agency or other pulls this hook (and together with it the box) with a constant force. To an observer outside, not being pulled, the box will appear to move with constant acceleration upwards, and finally acquire an enormous velocity. But how would the observer in the box interpret the state of affairs? He would have to use his legs to support himself and this would give him the sensation of weight. Objects which he is holding in his hands and releases will fall relatively to the floor with acceleration, for the acceleration of the box will no longer be communicated to them by the hand; moreover, all bodies will "fall" to the floor with the same acceleration. The observer in the box, whom we suppose to be familiar with gravitational fields, will conclude that he is situated in a uniform field of gravitation: the hook in the ceiling will lead him to suppose that the box is suspended at rest in the field and will account for the box not falling in the field. Now the interpretation of the observer in the box and the observer outside, who is not being pulled, are equally justifiable and valid, as long as the equality of inertial and gravitational mass is maintained.

We may now enunciate **Einstein's Principle of Equivalence:** Any change which an observer perceives in the passage of an event to be due to a gravitational field would be perceived by him exactly in the same way, if the gravitational field were not present and provided that he—the ob-

server—make his system of reference move with the acceleration which was characteristic of the gravitation at his point of observation.

It might be concluded from this that one can always choose a rigid body of reference such that, with respect to it, no gravitational field exists, i.e. the gravitational field may be eliminated; this, however, only holds for particular cases. It would be impossible, for example, to choose a rigid body of reference such that the gravitational field of the earth with respect to it vanishes entirely.

The principle of equivalence enables us theoretically to deduce the influence of a gravitational field on events, the laws of which are known for the special case in which the gravitational field is absent.

We are familiar with space-time-domains, which are approximately Galilean when referred to an appropriate rigid body of reference. If we refer such a domain to a rigid body of reference K^1 moving irregularly in any arbitrary fashion, we may assume that a gravitational field varying both with respect to time and to space is present for K^1 : the nature of this field depends on the choice of the motion of K^1 . This enabled Einstein to discover the laws which a gravitational field itself satisfies. It is important to notice that Einstein does not seek to build up a model to explain gravitation but merely proposes a *theory of motions*. His equations describe the motion of any body in terms of co-ordinates of the space-time manifold, making use of the interchangeability and equivalence implied in relativity. He does not discuss forces *as such*; they are, after all, as Karl Pearson states, "arbitrary conceptual measures of motion without any perceptual equivalent." They are simply intermediaries which have been inserted between matter and motion from analogy with our muscular sense.

A direct consequence of the application of the Principle of Equivalence in its general form is that the velocity of light varies for different gravitational fields, and is constant only for uniform fields (this does not contradict the special theory of relativity, which was built up for uniform fields, and only makes it a special case of this much more general theory of relativity). But change of velocity implies refraction, i.e. a ray of light must have a curved path in passing through a variable field of gravitation. This affords a very valuable test of the truth of the theory, since a star, the rays from which pass very near the sun before reaching us, would have to appear displaced (owing to the stronger gravitational field around the sun), in comparison with its relative position when the sun is in another part of the heavens : this effect can only be investigated during a total eclipse of the sun, when its light does not overpower the rays passing close to it from the star in question.* The calculated curvature is, of course, exceedingly small (1.7 seconds of arc), but, nevertheless, should be observable.

The motion of the perihelion of Mercury, discovered by Leverrier, which long proved an insuperable obstacle regarded in the light of Newtonian mechanics, is immediately accounted for by the general theory of relativity ; this is a very remarkable confirmation of the theory.

Before we finally enunciate the general theory of relativity, it is necessary to consider a special form of acceleration, viz. rotation. Let us suppose a space-time-domain (referred to a rigid body K) in which the first Newtonian Law holds, i.e. a Galilean field : we shall suppose a second rigid body of reference K^1 to be rotating uniformly with respect to K ,

* We shall return to this test at the conclusion of the chapter.

say a plane disc rotating in its plane with constant angular velocity. An observer situated on the disc near its periphery will experience a force radially outwards, which is interpreted by an external observer at rest relatively to K as centrifugal force, due to the inertia of the rotating observer. But according to the principle of equivalence the rotating observer is justified in assuming himself to be at rest, i.e. the disc to be at rest. He regards the force acting on him as an effect of a particular sort of gravitational field (in which the field vanishes at the centre and increases as the distance from the centre outwards). This rotating observer, who considers himself at rest, now performs experiments with clocks and measuring-scales in order to be able to define time- and space-data with reference to K¹. It is easy to show that if, of two clocks which go at exactly the same rate when relatively at rest in the Galilean field K, one be placed at the centre of the rotating disc and one at the circumference, the latter will continually lose time as compared with the former.

Secondly, if an observer at rest in K measure the radius and circumference of the rotating disc, he will obtain the same value for the radius as when the disc is at rest, but since, when he measures the circumference of the disc, the scale lies along the direction of motion, it suffers contraction, and, consequently, will divide more often into the circumference than if the scale and the disc were at rest. (The circumference does not change, of course, in rotation.) That is, he would get a value greater than π for the ratio $\frac{\text{circumference}}{\text{diameter}}$. This means that

Euclidean geometry does not hold for the observer making his observations on the disc, and we are obliged to use co-ordinates which will enable his

results to be expressed consistently. Gauss invented a method for the mathematical treatment of any continua whatsoever, in which measure-relations ("distance" of neighbouring points) are defined. Just as many numbers (Gaussian or curvilinear co-ordinates) are assigned to each point as the continuum has dimensions. The allocation of numbers is such that the uniqueness of each point is preserved and that numbers whose difference is infinitely small are assigned to infinitely near points. This Gaussian or curvilinear system of co-ordinates is a logical generalization of the Cartesian system. It has the great advantage of also being applicable to non-Euclidean continua, but only in the cases in which infinitesimal portions of the continuum considered are of the Euclidean form. This calls to mind the remarks made at the commencement of this sketch about the validity of geometrical theorems. It seems as though the miniature view that we can take of straight lines in the immensity of space led to a firm belief in the universal significance of Euclidean geometry. When we deal with light phenomena which range to enormous distances, we find that we are not justified in confining ourselves to Euclidean geometry; the "straightest" line in the time-space-manifold is "curved." We must therefore choose that geometry which, expressed analytically, enables us to describe observed phenomena most simply: it is clear that for even large finite portions of space the non-Euclidean geometry chosen must practically coincide with Euclidean geometry.

We now see that the general theory of relativity cannot admit that *all rigid* bodies of reference K, K^1 , etc., are equally justifiable for the description of the general laws underlying the phenomena of physical nature, since it is, in general, not possible to make

use of rigid bodies of reference for space-time descriptions of events in the manner of the special theory of relativity. Using Gaussian co-ordinates, i.e. labelling each point in space with four arbitrary numbers in the way specified above (three of these correspond to three space dimensions and one to time), the general principle of relativity may be enunciated thus :—

All Gaussian four-dimensional co-ordinate-systems are equally applicable for formulating the general laws of physics. This carries the principle of relativity, i.e. of equivalence of systems, to an extreme limit.

With regard to the relativity of rotations, it may be briefly mentioned that centrifugal forces can, according to the general theory of relativity, be due only to the presence of other bodies. This will be better understood by imagining an isolated body poised in space ; there could be no meaning in saying that it rotated, for there would be nothing to which such a rotation could be referred : classical mechanics however, asserts that, in spite of the absence of other bodies, centrifugal forces would manifest themselves : this is denied by the general theory of relativity. No experimental test has hitherto been devised which could be carried out practically to give a decision in favour of either theory.

A favourable opportunity for detecting the slight curvature of light rays (which is predicted by the general theory) when passing in close vicinity to the sun occurred during the total eclipse of the 29th May, 1919. The results, which were made public at the meeting of the Royal Society on 6th November following, were reported as confirming the theory.

In addition to the slight motion of Mercury's perihelion, there is still a third test which is based upon a shift of the spectral line towards the infra-red, as

a result of an application of Doppler's principle ; this has not yet led to a conclusive experimental result.

I. NOTE ON NON-EUCLIDEAN GEOMETRY

In practical geometry we do not actually deal with straight lines, but only with distances, i.e. with finite parts of straight lines, yet we feel irresistibly impelled to form some conception of the parts of a straight line which vanish into inconceivably distant regions. We are accustomed to imagining that a straight line may be produced to an infinite distance in either direction, yet in our mathematical reasoning we find that in order to preserve consistency (in Euclid),* we may only allocate to this straight line *one* point at infinity : we say that two straight lines are parallel when they cut at a point at infinity, i.e. this point is at an infinite distance from an arbitrary starting-point on either straight line, and is reached by moving forwards or backwards on *either*.

Many attempts have been made, without success, to deduce Euclid's "axiom of parallels," which asserts that only one straight line can be drawn parallel to another straight line through a point outside the latter, from the other axioms. It finally came to be recognized that this axiom of parallels was an unnecessary assumption, and that one could quite well build up other geometries by making other equally justified assumptions.

If we consider a point, P , outside a straight line, L (Fig. 4), to send out rays in all directions, then, starting from the perpendicular position Pa_1 , we find that the more obliquely the ray falls on L the farther does the point of intersection a_n travel

* According to the modern analytical interpretation of Euclid.

along L to the left (say). Our experience teaches us that the ray and L have one point in common. There is no justifiable reason, however, for asserting, as Euclid's axiom does, that for a final infinitely small increase of the angle $a_1 Pa_n$ (i.e. additional turn of Pa_n about P), a_n suddenly bounds off to infinity along L , i.e. a_n , the point of intersection leaves finite regions to disappear into so-called "infinity," and that, for a further infinitesimal increase, a_n , reappears at infinity at the other end of L to the right of a_1 .

One might equally well assume, as Lobatschewsky did, that Pa_∞ and $Pa_{-\infty}$ form an angle which differs ever so slightly from two right angles, and that there are an infinite number of other straight lines included

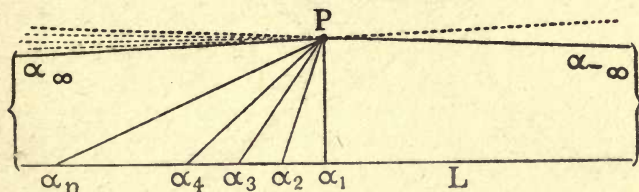


FIG. 4.

between these two positions (as indicated by the dotted lines in the figure), which do not cut L at all. Lobatschewsky (and also Bolyai) built up an entirely consistent geometry on this latter assumption.

Riemann later abolished the assumption of infinite length of a straight line, and assumed that in traveling along a straight line sufficiently far one finally arrives at the starting-point again without having encountered any limit or barrier. This means that our space is regarded as being finite but *unbounded*.*

* E.g. the surface of a sphere cuts a finite volume out of space, but particles sliding on the surface nowhere encounter boundaries or barriers. This is a three-dimensional analogon to the four-

Thus in Riemann's case there is *no* parallel line to L for a_n never leaves L ; there is no a_∞ . This geometry was called by Klein elliptical geometry (and includes spherical geometry as a special case). He calls Euclidean geometry parabolic (Fig. 5), for the branches of a parabola continue to recede from one to another, and yet in order to obtain consistent results in its formulæ we are obliged only to assign one point at infinity to it, just as to the Euclidean straight line. Lobatschewsky's geometry is similarly called hyperbolic (Fig. 5), since a hyperbola has

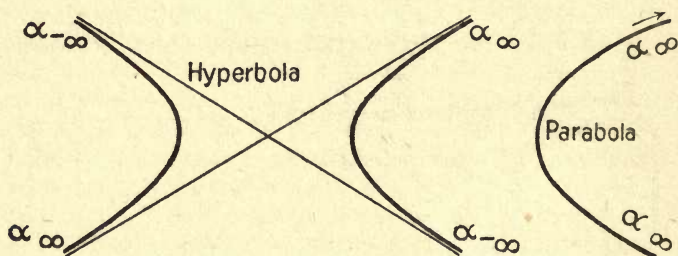


FIG. 5.

two points at infinity, corresponding in analogy to the two points at infinity at which the two parallels through a point external to a straight line cut the latter.

The fact that one is obliged to renounce Euclidean geometry in the general theory of relativity leads to the conclusion that our space is to be regarded as finite but unbounded: it is curved, as Einstein expresses it, like the faintest of ripples on a surface of water.

dimensional space-time manifold of Minkowski. It does not mean that the universe is enclosed by a spherical shell, as was supposed by the ancients. We cannot form a *picture* of the corresponding result in the four-dimensional continuum in which, according to the general theory of relativity, we live.

SOME ASPECTS OF RELATIVITY

THE THIRD TEST

BY HENRY L. BROSE, M.A.

UP to the present, three methods of verifying Einstein's Theory of Relativity have been suggested.

The first one, which was a direct outcome of the new gravitational field-equations proposed by Einstein, proved successful. The slow motion of Mercury's perihelion which long mystified astronomers was immediately accounted for. This result is the more remarkable as all other explanations of this phenomena were artificial in origin, consisting of a hypothesis formulated *ad hoc* which could not be verified by observation.

The second method involved the deflection of a ray of light in its passage through a varying gravitational field. The results of the total eclipse of the sun which occurred on 29th May, 1919, have become famous and were recorded as confirming Einstein's prediction. The results of a more recent expedition have proved finally conclusive.

The third test, the results of which are still in abeyance, is perhaps the most important of the three, inasmuch as it depends upon a very simple calculation from Einstein's Principle of Equivalence, which asserts that an observer cannot discriminate how

much of his motion is due to a gravitational field and how much is due to an acceleration of his body of reference. Einstein illustrates his argument by supposing an observer situated in a closed box in free space. The observer has at first no sensation of weight, and need not support himself upon his feet. Now suppose an external agent to pull the box in a definite direction with constant force. The observer in the box performs experiments with masses of variable material, and as they all fall to the "floor" of the box at the same time, he concludes that he is in a gravitational field. He himself has acquired the sensation of weight. This result led Einstein to propound the equivalence of gravitational and accelerational fields. An immediate consequence of this principle is that the duration of an event depends upon the gravitational conditions at the place of the event.

If we consider the light (of frequency ν_1) which is emitted by a distant star, and suppose it to traverse a practically invariable gravitational field in which bodies are assumed to fall with a constant acceleration g , then an observer at a distance h from the star will have attained a velocity ($=$ acceleration $\times$ time) $= g \cdot \frac{h}{c}$ where c is the velocity of light and the distance h is small in comparison with the distance traversed by the observer in the time the light takes to reach him. By Doppler's Principle, the apparent frequency ν_2 is given by

$$\nu_2 = \nu_1 \left(1 + \frac{v}{c} \right) = \nu_1 \left(1 + \frac{g \cdot \frac{h}{c}}{c} \right) = \nu_1 \left(1 + \frac{gh}{c^2} \right).$$

Potential of unit mass moved through a distance h is $gh = \pm \phi$ (say). This gives the work done in moving unit mass from the source of light to the

observer (the source of light is here the point to which the potential energy is referred in the field).

Therefore, if we transform the accelerational field of the observer into the gravitational field, we get the result :

$$\nu_2 = \nu_1 \left(1 \pm \frac{\phi}{c^2} \right).$$

This means that a spectral line of frequency ν_1 will appear to a distant observer to be displaced, if compared with the position of the same line, when produced by a source at a different point in the field. Each of these lines, produced by vibrating electrons, may be regarded as a clock, and this simple calculation shows how time-measurements are affected by the state of the gravitational field. This effect amounts to 0.008 Ångströms, for a wavelength of 4000 Å. The same displacement would be produced as a Doppler effect by a velocity of 0.6 kms. per sec. When this test was put into practice, it was found difficult to discriminate it from the various superposed effects due to other causes such as the radial velocities of the stars, proper velocities of the gaseous envelopes, pressure, etc. The conditions of the emission of light by the sun have not been fully ascertained, nor is the light of the arc lamp free from disturbing elements. Dr. Erwin Freundlich, of the Neubabelsberg Observatory, has discussed, in conjunction with Professor Einstein, the possibility of recognizing this effect in spite of these obscuring influences. He points out three ways of establishing the result qualitatively. They may be briefly classified as being based on (1) statistical methods; (2) nebular spectra; (3) calcium lines in the spectra of the atmosphere surrounding double-stars.

1. If we consider a great number of stars of about

the same mass evenly distributed over the heavens, and represent the spectral shift due to radial velocities (i.e. velocities in the line of sight) graphically, we should expect these velocities to be distributed according to the law of probability about the value zero, i.e. as depicted by Gauss's Error Curve, which resembles a vertical section of a bell. If, however, Einstein's gravitational effect really exists, we should expect these velocities to group themselves symmetrically about a positive velocity which would be that corresponding to this spectral shift. Gauss's Error Curve would thus appear displaced by precisely the amount of the radial velocity corresponding to this shift, as all the radial velocities would be falsified by just this amount.

The values of the radial velocities have been plotted in the case of B-stars, called Helium stars on account of the predominance of helium lines in their spectra. Other observations have led astronomers to infer that the B-stars have unusually great masses but small densities. The result has been distinctly in favour of the Einstein shift on the basis of the foregoing discussion. The same was found to hold for the bright K- and M-stars, which are considered to be at a lower temperature and possessed of enormous surface extent, which accounts for their brilliance.

If we indicate the mean shift of the lines towards the red by K , then for

$$\begin{array}{lll} \text{B-stars } K = 4.3 \text{ kms. } \pm 0.5, \\ K \text{ ,, } K = 3.5 \text{ ,, } \pm 0.9, \\ M \text{ ,, } K = 5.3 \text{ ,, } \pm 2.3. \end{array}$$

K is here expressed in terms of a Doppler shift as a velocity, i.e. as if the Einstein shift were due to an additional radial motion and hence expressible in kilometres.

Alternative ways of accounting for this shift have been proposed.

(a) It may be regarded as an ordinary Doppler effect. This would imply that the stars of the B, K, and M type suffer a general expansion to which stars of the F and G type (yellow stars like the sun) and the A-stars are not subject.

This explanation does not seem very probable, as helium lines were used in determining the shift for the B-stars, whereas quite different lines were used for measuring the effect for the K- and M-stars. It would be a strange coincidence if this shift, to which all the evidence points as arising from a common origin, should be manifested just in these cases which have been made the object of an investigation.

(b) The general shift towards the red might be ascribed to pressure effects at the surfaces of the stars or to the presence of other lines which lie on the red side of the main lines, but which are very weak or even absent in the comparison spectrum of the sun. A detailed knowledge of conditions on the surfaces of stellar bodies could alone give a decision on this point.

2. It is only possible to prove that the shift K is not due to a radial velocity if one can measure the ordinary Doppler effect arising from the radial velocity separately. Let us consider a single B-star or group of B-stars which happen to be embedded in a nebula of great extent which accompanies them in their motion. The Doppler effect due to the radial velocity would be the same for the star as the nebula, but the gravitational effect predicted by Einstein would not be the same, inasmuch as the gravitational field at the surface of the star will vary considerably from that at the outer edge of the nebula. Hence it would be reasonable to attribute

any difference in the magnitude of the spectral shifts in the case of the star and the nebula to the difference in gravitational fields at each place.

The stars of the nebular group of Orion have hitherto offered the only possibility of applying this method. The results have fulfilled Einstein's expectations qualitatively, and it remains to be seen whether the agreement will hold quantitatively. A general shift of the star-spectrum as compared with the corresponding lines of the associated nebula was observed.

Some very bright B-stars in the constellation of Orion are considered to form an entity with their attendant nebula. This conclusion was reached as the result of independent research.

The radial velocity of the Orion-nebula has been measured by various observers. The values obtained are: 17.7 (Wright), 17.4 (Vogel and Eberhardt), 18.5 (Frost and Adams). The mean value is 17.4 kms. per sec. This velocity is derived from the brightest part of the nebula, the so-called trapezium. The values obtained in the case of the stars almost all exceed 20 kms. per sec., and hence it seems likely that part of this radial velocity, viz. the excess over that of the nebula, is due to the Einstein effect. When the difference between the radial velocities of the stars and the associated nebula are tabulated for each star, we find that in the case of all members except two the difference is positive, i.e. indicative of a shift towards the red end, in agreement with the statistical investigation applied to the B-, K-, and M-stars. The difference amounts to 6.0 kms. ± 1 km., and is a little greater than that given by the statistical method.

The two stars ψ and 36 Orionis give a displacement towards the violet end. It has been suggested that they do not belong to the more limited group of

Orion stars, but are only projected into that portion of the celestial sphere. This is supported by the fact that both stars have only very small spherical proper motions, and that the radial velocities observed for them differ considerably from the mean of the radial velocities of the others.

This method has not been successfully applied to other stellar systems inasmuch as the nebulae of those which are available emit such feeble light that it has not been possible to establish the displacement to any degree of accuracy. Eddington recently pointed out that a very important factor had been neglected in the fundamental equations of the early theories concerning the equilibrium of stellar matter, viz. the pressure due to radiation. According to his theory, the equilibrium in the interior of the star (regarded as a gaseous sphere) is determined by three conditions. These are gaseous pressure, radiational pressure, and gravitational forces.

Calculation shows that for very great masses the gravitational pressure is almost entirely balanced by radiational pressure. This implies that any additional force such as that due to a centrifugal field of rotation would lead to an unstable condition.

It can, furthermore, be deduced from Eddington's theory that only stars whose masses exceed a certain minimum value can in the course of their evolution reach the very high surface-temperatures which have been observed in the case of the O- and B-stars.

It therefore seems likely that the O- and B-stars have in the process of evolution passed through a stage of which the radiational pressure has brought about a condition of unstable equilibrium, and one might expect them to be surrounded by cosmic dust which has become dissociated from the nuclei of the system.

In some cases this dissociated matter may be in

a very fine state of division, and may extend so far into space that the absorption lines they produce in the spectrum of the star they surround may originate from a gravitational field which differs perceptibly from that at the surface of the star. There are definite signs of the existence of such atmospheres. A high percentage of B-stars are found to be spectroscopic double stars, i.e. their spectral lines fluctuate periodically about some mean position. Hartmann was the first to notice that in the spectrum of the B-star S. Orionis the absorption lines K and H of calcium, viz. 3933.82 and 3968.63 Ångströms, occur, but that they do not share in the periodic movements of the other lines. A number of other stars belonging to early spectral types contain calcium absorption lines in their spectra, which exhibit a similar anomaly, inasmuch as they either remain immovable or execute periodic motions which are of feeble amplitude compared with the proper stellar lines. In view of the important rôle that calcium plays in the outermost layers of the gaseous atmosphere encircling the sun, and in view of the discussion above, the suggestion forces itself upon one that these calcium lines indicate the presence of an extensive atmosphere surrounding the star.

It has often been put forward that these lines are due to the light from these stars being absorbed by vast interstellar clouds of calcium. Evershed considers that this is supported by the fact that when the motion of the solar system is subtracted from that calculated from the fixed calcium lines (owing to the ordinary Doppler effect), the remaining motion is very small. But this argument does not carry weight inasmuch as it is known that the B-stars, in the spectrum of which these lines occur, themselves have very small radial velocities. As Young remarked, it seems very strange that these calcium

clouds should so consistently choose to lie in front of stars of type B or earlier. An objection against this hypothesis is to be found in the fact that in the case of various systems these two calcium lines are not at rest but move, although with somewhat less amplitude than the other proper lines of the double star.

An additional circumstance which lends support to the theory that calcium lines denote the presence of an atmosphere around the star is that a great number of helium-stars are enveloped in a nebulous atmosphere which is actually visible.

Assuming then that the calcium absorption lines are due to such atmospheres, we may apply the same process as in the case of the Orion nebula, i.e. if the shifts of the spectral-lines of the stars be systematically falsified by a superposed gravitational effect, this should be expressed by the lines of the actual spectrum from a double star being displaced towards the red as compared with the fixed calcium lines.

This phenomenon has been clearly observed. The result has not yet been *quantitatively* fixed, as the numbers taken are not regarded as final.

All stars in the spectra of which the H and K lines of calcium occur have been used to test the conclusion, and all show a shift to the red end; the mean of the shifts corresponds to a velocity of $+6.3$ kms. per sec.

The results of this discussion have been formulated by Dr. Freundlich thus :—

SUMMARY

I. Statistical consideration gives us the means of separating the mean gravitational effect from the ordinary Doppler effect in the case of the helium B-stars and the bright K- and M-stars, which

astronomical investigations compel us to regard as being of particularly great mass.

A general shift of the spectra towards the red is exhibited with considerable certainty.

2. It follows from a comparison of the displacement of the lines of the star-spectra that the above displacement which was found by a statistical examination is not an ordinary Doppler effect, but is due to the conditions of emission of light at the surfaces of the stars.

3. The close connection of the B- and O-stars with nebulous matter in the heavens is a symptom that these stars are of great mass.

4. If we regard the fixed calcium lines in the spectra of B- and O-stars as being caused by absorption in extended calcium atmospheres moving with each star in question, the shift towards the red which manifests itself may be regarded as the effect predicted by Einstein's theory, i.e. due to the different gravitational fields from which the absorption lines and the stellar lines have originated.

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